THE PICK INTERPOLATION THEOREM FOR FINITELY CONNECTED DOMAINS

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Let D be the open unit disk, let $z_1, \ldots, z_n$ be distinct points in D, and let $w_1, \ldots, w_n$ be complex numbers. A theorem of Pick asserts that there is an analytic function ϕ on D satisfying $|\phi(z)| \leq 1$ for z in D and $\phi(z_i) = w_i$ for $i = 1, \ldots, n$ if and only if the matrix

$$\left[\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\right]$$

is nonnegative (positive semidefinite); moreover, the interpolating function ϕ is unique if and only if the determinant of this matrix is zero [16]. The purpose of this paper is to generalize this theorem with D replaced by a finitely connected domain in the plane.

To state the general result, let R be a bounded domain in the plane whose boundary consists of $p + 1$ disjoint analytic Jordan curves, let ∂R denote the boundary of R, let ρ be a nonnegative Borel measurable function on ∂R which is bounded and bounded away from zero, let μ be the measure $d\mu(z) = \rho(z)\, d|z|$, and let $\Lambda = \{(\alpha_1, \ldots, \alpha_p): |\alpha_k| = 1 \text{ for } k = 1, \ldots, p\}$ be the p-torus. For α in Λ, there is a Hardy space $H^2_\alpha(R)$ of multiple-valued analytic functions on R which are modulus automorphic of index α. These spaces arise in questions on factorization [25], invariant subspaces [18], [23], [24], subnormal operators [2], and extremal polynomials [26]. The space $H^2_\alpha(R)$ can be viewed as a closed subspace of $L^2(\mu)$ and, using the norm in $L^2(\mu)$, the space $H^2_\alpha(R)$ is a functional Hilbert Space over R. Thus, there is a kernel function $k^\alpha(s,t)$ on $R \times R$ such that for f in $H^2_\alpha(R)$ $f(t) = \langle f, k^\alpha_t \rangle$ where $k^\alpha_t(s) = k^\alpha(s,t)$.

THEOREM. *Let* $z_1, \ldots, z_n$ *be distinct points in* R *and let* $w_1, \ldots, w_n$ *be complex numbers. There is an analytic function* ϕ *on* R *satisfying* $|\phi(z)| \leq 1$ *for* z *in* R *and* $\phi(z_i) = w_i$ *for* $i = 1, \ldots, n$ *if and only if the matrix*

$$[(1 - w_i \bar{w}_j)\, k^\alpha(z_i, z_j)]$$

is nonnegative for each α *in* Λ. *The interpolating function* ϕ *is unique if and only if the determinant of this matrix is zero for some* α.

Note that if R is the unit disk and if $\rho \equiv 1$, then Λ consists of one point and the one kernel function involved is the Szegö kernel $k(s,t) = (2\pi)^{-1}(1 - s\bar{t})^{-1}$.

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