ON THE CLASSIFICATION OF STATIONARY SETS

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1. INTRODUCTION

In this paper, a *cardinal* is an initial ordinal and a cardinal α is *regular* provided it is not the sum of fewer, smaller cardinals. We will reserve the symbol κ for a fixed regular uncountable cardinal. Viewing the ordinal κ as the set of smaller ordinals we will speak of "subsets of κ" instead of "subsets of $[0,\kappa)$." (However, see the comments on special notations at the end of this section.)

The set κ will always carry the usual order topology and subsets S of κ will always be endowed with the relative topology inherited from κ. We define

$$\operatorname{cub}(\kappa) = \{S \subset \kappa : S \text{ is closed and unbounded (equivalently, cofinal in } \kappa)\}$$

We say that a set $S \subset \kappa$ is *stationary* if $S \cap C \neq \emptyset$ for each $C \in \operatorname{cub}(\kappa)$ and that S is *bistationary* if both S and $\kappa - S$ are stationary.

Regular cardinals and their stationary subsets have long been important tools in topology, especially as indexing and constructive devices, *e.g.*, in the recent papers [4], [5], [7], [8]. They have received less attention as topological objects in their own right, except in the theory of ordered spaces where stationary sets in regular cardinals are viewed as the archetypical non-paracompact ordered spaces. More precisely, it is proved in [3] that if X is (a subspace of) a linearly ordered topological space, then X is *not* paracompact if and only if some closed subspace of X is homeomorphic to a stationary set in an uncountable regular cardinal. That theorem raises the question of whether two stationary subsets of a fixed κ can be distinguished topologically from each other. Phrased in that way, the question has an immediate affirmative answer since, if S is a bistationary set in κ, then no member of $\operatorname{cub}(\kappa)$ can even be mapped continuously onto a cofinal subset of S. (Further, the existence of bistationary sets is guaranteed by the Ulam-Solovay theorem cited as Theorem E below.) However it is more difficult to determine whether two bistationary sets in κ, *e.g.*, a bistationary set and its complement, are of different topological types, especially if $\kappa = \omega_1$. In this paper we settle that question by proving

A. THEOREM. *If* S *and* T *are disjoint bistationary subsets of* κ, *then there is no continuous mapping of* S *onto a cofinal subset of* T.

Indeed we show that there are many topologically incomparable types of stationary subsets of κ by proving

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