A NEW APPROACH TO GELFAND-MAZUR THEORY AND THE EXTENSION THEOREM

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To state the theorem on which our approach is based, we need the following definitions: A function N from a ring A to the real numbers $\mathbb{R}$ is a (ring) *seminorm* if for all $x, y \in A$, $N(x) \geq 0$, $N(-x) = N(x)$, $N(xy) \leq N(x)N(y)$, and $N(x+y) \leq N(x) + N(y)$. The null space $N^{-1}(0)$ of a seminorm N is an ideal; N is a *norm* if $N^{-1}(0) = (0)$. The *core* of a seminorm N on A is the set C(N), defined by

$$C(N) = \{c \in A: N(c) \neq 0, \text{ and } N(cx) = N(c)N(x) = N(xc) \text{ for all } x \in A\}.$$

A function V from a ring A with identity to $\mathbb{R}$ is an *absolute semivalue* if V is a seminorm satisfying $V(1) = 1$ and $V(xy) = V(x)V(y)$ for all $x, y \in A$. The null space $V^{-1}(0)$ of an absolute semivalue V is a prime ideal; V is an *absolute value* if $V^{-1}(0) = (0)$.

If $|\,.\,.\,|$ is an absolute value on a field K and if A is a K-algebra, a ring norm N on A is an *algebra norm* if $N(\lambda.x) = |\lambda|\, N(x)$ for all $\lambda \in K$, $x \in A$.

Let N be a seminorm on a commutative ring A. As is well known, $\lim_{n\to\infty} N(x^n)^{1/n}$ exists for each $x \in A$, and $N_s: x \mapsto \lim_{n\to\infty} N(x^n)^{1/n}$ is a seminorm on A, called the *spectral seminorm* associated to N. A seminorm N on A is *spectral* if $N = N_s$, or equivalently, if $N(x^n) = N(x)^n$ for all $x \in A$ and all $n \geq 1$. If N is any seminorm on A, N_s is a spectral seminorm.

Our discussion is based on the following theorem of Aurora [3, Theorem 1]:

THEOREM 1. *If* N *is a nonzero spectral seminorm on a commutative ring* A *with identity and if* $J = N^{-1}(0)$, *there is a family* $(V_c)_{c\in A\setminus J}$ *of absolute semivalues on* A *such that for each* $c \in A\setminus J$, $V_c \leq N$, $C(V_c) \supseteq C(N) \cup \{c\}$, $V_c(x) = N(x)$ *for all* $x \in C(N) \cup \{c\}$, *and therefore* $N = \sup_{c\in A\setminus J} V_c$.

We have stated somewhat more than appears in [3], but a slight modification of the proof yields our statement. Earlier, Cohn [8, Theorem 13.3] had shown that a spectral norm on a field was the supremum of a family of absolute values, but the further properties mentioned in Theorem 1, which are crucial for applications, are not apparently derivable from his proof. Aurora's theorem has subsequently been rediscovered, in whole by Bergman [4], and in part by Szpiro [17] and Kiyek [11].

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