MATRIX ALGEBRAS OVER O_n

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This paper is concerned with the extension theory of the C*-algebras O_n studied by J. Cuntz in [7] and their tensor products with the algebra M_k of complex $k \times k$ matrices. We show by computing various Ext groups that the O_n's are pairwise non-isomorphic (a result which has also been obtained independently by M. Pimsner and S. Popa [8]), and that O_n and $O_n \otimes M_k$ are non-isomorphic if k and $n-1$ are not relatively prime. We also prove that O_n is isomorphic to $O_n \otimes M_k$ if k divides n or is congruent to 1 mod$(n-1)$.

We briefly indicate our notation and summarize essential prerequisites from extension theory for C*-algebras. Throughout, H is complex, infinite-dimensional separable Hilbert space. We write L(H) and Q(H) for, respectively, the algebra of all bounded operators on H and the Calkin algebra (the quotient of L(H) by the compacts), and let π: L(H) $\to$ Q(H) denote the quotient map. To avoid unnecessary clumsiness of expression, we once and for all make fixed identifications of $H \otimes \mathbb{C}^n$ (the direct sum of n copies of H) with H for $n = 2, 3, \ldots$, and thereby identify $L(H \otimes \mathbb{C}^n)$ with L(H) and $Q(H \otimes \mathbb{C}^n)$ with Q(H). We also identify $L(H \otimes \mathbb{C}^n)$ and $Q(H \otimes \mathbb{C}^n)$ with $L(H) \otimes M_n$ and $Q(H) \otimes M_n$, respectively, in the natural way. For a separable unital C*-algebra A, we write E(A) for the set of all unital *-monomorphisms (*extensions*) of A into Q(H). We say that extensions τ and σ are *strongly* (respectively, *weakly*) *equivalent* if there is a unitary $U \in L(H)$ (respectively, unitary $u \in Q(H)$) such that $\tau(\cdot) = \pi(U)\sigma(\cdot)\pi(U^*)$ (resp. $u\sigma(\cdot)u^*$). For $\tau \in E(A)$, $[\tau]$ denotes the strong equivalence class of τ. We write $\mathrm{Ext}^s(A)$ for $\{[\tau] : \tau \in E(A)\}$ and let $\mathrm{Ext}^w(A)$ denote the set of weak equivalence classes in E(A). Given $\tau, \sigma \in E(A)$, we define $\tau \oplus \sigma \in E(A)$ (via our identification of Q(H) with $Q(H) \otimes M_2$) by

$$(\tau \oplus \sigma)(a) = \begin{pmatrix} \tau(a) & 0 \\ 0 & \sigma(a) \end{pmatrix}.$$

The operations thereby induced on $\mathrm{Ext}^s(A)$ and $\mathrm{Ext}^w(A)$ make them into abelian semigroups. An extension τ is called *trivial* if it lifts to a unital *-representation of A on H. D. Voiculescu showed in [12] (see also [2]) that all trivial extensions of A are strongly equivalent and that the resulting strong equivalence class serves as the zero element of $\mathrm{Ext}^s(A)$. Correspondingly, the weak equivalence class of any trivial extension is the zero element of $\mathrm{Ext}^w(A)$. It is not the case in general that $\mathrm{Ext}^s(A)$ (and hence $\mathrm{Ext}^w(A)$) is a group; see [1] for an example of a non-invertible extension. When $\mathrm{Ext}^s(A)$ is a group, though, $\mathrm{Ext}^w(A)$ can be naturally identified with the quotient of $\mathrm{Ext}^s(A)$ by the subgroup consisting of those $[\tau]$ for which τ is weakly equivalent to a trivial extension.

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