DISCRETE MAPS ON MANIFOLDS

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1. INTRODUCTION

Let M^n and N^n be second countable manifolds, and let $f: M^n \to N^n$ be a map (continuous function). The *branch set* $B_f \subset M^n$ is the set of points at which f fails to be a local homeomorphism; and f is *countable* (respectively, *discrete*) if $f^{-1}(y)$ is countable (respectively, consists of isolated points) for each $y \in N^n$.

1. THEOREM. *If* f *is countable, then* $\operatorname{int} B_f = \emptyset$, i.e. $\dim B_f \leq n - 1$.

2. THEOREM. *If* f *is discrete, then* $\dim B_f = \dim f(B_f) \leq n - 1$. *Specifically,* f *is open if and only if* $\dim B_f = \dim f(B_f) \leq n - 2$.

In [13] Väisälä proved Theorem 1 for $n \leq 3$, and conjectured it for general n. The present proof for arbitrary n is shorter than Väisälä's proof, but builds on his earlier lemmas, and his clever ideas. The second sentence of Theorem 2 is already known (see (12)). Examples ((9) and (10)) show that the Theorems are sharp, and a mistake in a paper of Trohimčuk [11] is discussed. The author is grateful to the Institute for Advanced Study for its hospitality during the summer of 1977.

3. *Notation and terminology.* A map $f: M^n \to N^n$ is *light* if

$$\dim f^{-1}(y) \leq 0 \qquad \text{for every } y \in N^n.$$

Alexander-Spanier cohomology with integer coefficients and compact supports is used, and $\check{H}^m$ is augmented. The real numbers are denoted by $\mathbb{R}$, $[0, 1] \subset \mathbb{R}$ by I, the unit sphere in $\mathbb{R}^{n+1}$ by S^n, and the distance between x and y by $d(x, y)$. A subset $A \subset B$ is *residual* if $B - A$ is of the first category in $B \neq \emptyset$ [8].

2. THE PROOF OF THEOREM 1

4. LEMMA. *Let* $K \neq \emptyset$ *be compact and let* $B \subset I^m$ *be residual* ($m = 0, 1, \ldots$). *If* $f: K \to I^m$ *is a light map with* $f \mid f^{-1}(B)$ *injective, then* $\check{H}^m(K) = 0$.

Proof. We use induction on m. For $m = 0$, I^0 is a single point, so $B = I^0$ and (since $f \mid f^{-1}(B)$ is injective), K is also a single point.

Suppose the lemma is true for $m - 1$, and consider $m \geq 1$. According to the Kuratowski-Ulam Theorem [8; Vol. I, p. 247, Corollary 1a] there is a residual set $C \subset I$ such that $B \cap (\{x\} \times I^{m-1})$ is residual in $\{x\} \times I^{m-1}$ for each $x \in C$.

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