SOME HIGHER ORDER DIFFERENCE SCHEMES ENFORCING AN ENTROPY INEQUALITY

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1. INTRODUCTION

Weak solutions of the initial value problem for hyperbolic systems of conservation laws,

$$(1.1) \qquad u_t + f(u)_x = 0, \qquad -\infty < x < \infty, t > 0; u(x, 0) \text{ given,}$$

are in general not unique. An entropy condition is imposed to select the physically relevant weak solution. It is clearly desirable to have such an entropy condition reflected in numerical methods for this type of problem; *i.e.,* to know that the limit of approximate solutions, when it exists, is the physically relevant solution.

In this context, the best understood methods are the "monotone" difference schemes. For these schemes, such a result is obtained in [3] for the case that (1.1) is a single equation, with u,f scalar valued. For the special example of Friedrichs' scheme, similar results for systems are also known [6]. Monotone schemes are also attractive in that they admit discrete representations of shock waves, at least for single equations [4]. Unfortunately, they are limited to first order accuracy, and thus are of limited practical importance.

In this paper we discuss some higher order schemes which also enforce an entropy condition. Our main results are for a second order scheme of Lax-Wendroff type. In the following, we shall assume either that (1.1) is a single equation or a system of dimension m with f_u symmetric. Smooth solutions of such systems satisfy an additional conservation law,

$$(1.2) \qquad U_t + F_x = 0,$$

in which U, F are scalar valued functions given by

$$(1.3) \qquad U = u^2, \quad F = 2(u \cdot f - \Phi)$$

where Φ is a scalar valued function satisfying $\Phi_u = f$. Discontinuous solutions of (1.1) do not satisfy (1.2). We shall, however, require such solutions to satisfy an entropy condition of the form

$$(1.4) \qquad U_t + F_x \le 0$$

in the sense of distributions. For systems which are strongly nonlinear in the sense of [10], this condition is equivalent to the classical entropy condition of

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