LOCAL EXTENSION OF CR FUNCTIONS FROM WEAKLY PSEUDOCONVEX BOUNDARIES

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Let $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$ be a domain in $\mathbb{C}^n$, $r \in C^2(\mathbb{C}^n)$, $dr \neq 0$ on $\partial\Omega$, and let $\bar\partial_b$ denote the tangential Cauchy-Riemann equations on $\partial\Omega$. A CR-function f on $\partial\Omega$ is a solution of $\bar\partial_b f = 0$; the exact sense in which this equation is interpreted may vary with the regularity of f and $\partial\Omega$. A basic result concerning CR-functions is the following local extension phenomenon, which holds at any strongly pseudoconvex point $p \in \partial\Omega$:

(*) *for each neighborhood* $U' \subset \mathbb{C}^n$ *of* p, *there exists a neighborhood* U'' *of* p *such that each* CR-*function* f *on* $\partial\Omega \cap U'$ *has a holomorphic extension to* $\Omega \cap U''$

(see the references in the survey article by Henkin and Chirka [2]). An important factor in the proof of (*) is that a strongly pseudoconvex boundary can be made (locally) strictly convex by a holomorphic change of coordinates. It is therefore immediate that (*) holds for $f \in \mathcal{O}(\partial\Omega \cap U')$. This local convexity is not true for weakly pseudoconvex domains (see Kohn and Nirenberg [3]), and the proof of (*) in this case is more delicate. Hill and MacKichan [1] have shown that (*) holds for the Kohn-Nirenberg example; they construct a family of disks rather differently from the way it is done below.

THEOREM. *Let* Ω *be a domain in* $\mathbb{C}^n$ *which is real analytic and (weakly) pseudoconvex in a neighborhood of* $p \in \partial\Omega$. *Then* (*) *holds at* p *if and only if there is no germ of a complex variety* V *of codimension one with* $p \in V \subset \partial\Omega$.

Proof. Let us first show that if (*) holds there can exist no germ of a complex hypersurface $V \subset \partial\Omega$. The condition that V has codimension one means that its normal bundle is given by $\partial r \wedge \bar\partial r$ and so it is a manifold. Thus there exists a function f holomorphic in a neighborhood of p such that $\{f = 0\}$ defines V at p and $d\,\mathrm{Re}\, f(p) = dr(p)$. A suitable branch of the function $F(z) = \exp(-f(z)^{-1/2})$ will define a C^∞, CR-function on a neighborhood of p in $\partial\Omega$ which cannot be continued to $\Omega \cap U''$ for any neighborhood U'' of p.

Now we show that (*) holds if V does not exist. More precisely, we will obtain a family of disks satisfying (i) and (ii) below which can be used to construct the extension. (A modern treatment of this is given, for instance, in Polking and Wells [4].) The proof that the function f can actually be extended can be carried

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