SINGULARITY OBSTRUCTIONS TO IMMERSIONS

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1. INTRODUCTION

Let M^m, N^n be connected C^∞ manifolds of dimension m and n, respectively, with $m < n$ and M^m closed (compact, without boundary). Let $f: M^m \to N^n$ be a continuous mapping. Our initial purpose here is to introduce three sets of (cohomological) homotopy invariants for f and show they are obstructions to the process of deforming f to a smooth (C^∞) immersion $g: M^m \to N^n$. By Smale-Hirsch theory [5] any obstruction to such a deformation must involve finding a vector bundle embedding $\phi: TM \to TN$ covering f, and in fact the vanishing of these invariants for an immersion is derived as a consequence of a more general proposition concerning the existence of bundle epimorphisms $\psi: \xi^n \to \tau^p$, ($\xi^n \to M^m$, $\tau^p \to M^m$ real vector bundles over M^m of rank n, p, respectively, $n > p$). Roughly speaking, our approach to this problem involves decomposing the vector bundle $\mathrm{Hom}(\xi, \tau) \to M^m$ into its "singularity subbundles" $S_i(\xi,\tau)$, $0 \leq i \leq p$, and asking when a section

$$\sigma \in \Gamma^\infty(\mathrm{Hom}(\xi, \tau))$$

may be homotoped so as to avoid all the $S_i(\xi,\tau)$, $i > 0$. Using the (Poincare duals of the) fundamental homology classes associated to each $\overline{S_i(\xi,\tau)}$ [13], obstructions to such a deformation are defined.

In Sections 2.1 and 2.2 we give the necessary background information on the homological properties of the $S_i(\xi,\tau)$ and prove the Proposition on bundle epimorphisms [Proposition 2.2.2]. The basic obstruction theorem is then given in (2.3) [Theorem 2.3.2].

In Section 3, we use the obstruction theorem together with a result due to R. Thom and I. Porteous [11] to study immersions of M^m into $\mathbb{C}P^n$, complex projective space of 2n real dimensions. The starting point here is the fact, which follows from the well-known theorem of A. Haefliger ([3], Theorem 1, p. 109), that every $f: M^m \to \mathbb{C}P^m$ deforms to a smooth immersion. The main result of this section [Theorem 3.1.2] then, deals with the question of when this result may be improved and when it is best possible.

Apart from its intrinsic interest, there is a second motivation for considering C^∞ immersions into $\mathbb{C}P^n$. Namely, in [6], A. Holme raised the question of computing the minimal dimension $n(V^m)$ for which V^m, a non-singular projective m-variety (over $\mathbb{C}$) may be embedded holomorphically in $\mathbb{C}P^n$. Since a negative result for C^∞ embeddings into $\mathbb{C}P^n$ is necessarily one for holomorphic embeddings (for M^m complex) the results of (3.1) carry over to give information on this problem. Pursuing

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