ON SOME CLASS NUMBER RELATIONS OF ALGEBRAIC TORI

Jih-Min Shyr

1. INTRODUCTION

A formula has long been known for expressing the class number h of a finite algebraic number field k in terms of other arithmetic invariants; namely

$$h = \frac{w|d|^{1/2}\rho}{2^r(2\pi)^t R}, \tag{1}$$

where w is the number of roots of unity contained in k, d is the discriminant of k, ρ is the residue at $s = 1$ of the Dedekind zeta function of k, R is the regulator of k, and r (resp. 2t) is the number of real (resp. complex) imbeddings of k into $\mathbb{C}$. This is equivalent to the fact that the volume of the compact group $k_A^1/k^\times$ is equal to ρ, where k_A^1 and $k^\times$ are the group of k-idele with volume 1 and the group of nonzero elements of k, respectively (*cf.* [9]). Or, in the language of algebraic groups, (1) is equivalent to the fact that the Tamagawa number of the algebraic torus $R_{k/\mathbb{Q}}(G_m)$ over $\mathbb{Q}$ is 1 (*cf.* [6]), $R_{k/\mathbb{Q}}$ being the Weil functor of restricting the field of definition from k to $\mathbb{Q}$ (*cf.* [10]). In view of the above interpretation, we shall generalize (1) to a formula for the class number of an algebraic torus T defined over $\mathbb{Q}$ which has the same form as (1) except a factor involving the Tamagawa number of T. From this generalized class number formula, we obtain a relation which expresses the relative class number of two isogenous tori in terms of their Tamagawa numbers and certain indices of the maps induced naturally by an isogeny between them. We shall also indicate how the above-mentioned class number relation can be applied to the studies of totally positive binary quadratic forms over totally real algebraic number fields and norm forms of algebraic number fields. The details will be discussed elsewhere.

In this paper, we shall use the following standard notations: $\mathbb{Z}$ for the ring of rational integers; $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ for the fields of rational, real, and complex numbers; $\mathbb{R}_+^\times$ for the multiplicative group of positive real numbers; Ω for a universal domain containing $\mathbb{Q}$; $R^\times$ for the multiplicative group of invertible elements of a ring R; [G] for the order of a group G; and Ker α, Im α, and Cok α for the kernel, image, and cokernel of a homomorphism α.

Let G, G' be commutative groups, α a homomorphism $G \to G'$. If Ker α, Cok α are both finite, we define the q-symbol of α by $q(\alpha) = [\text{Cok}\ \alpha]/[\text{Ker}\ \alpha]$.

2. PRELIMINARIES

We shall start by recalling some basic definitions and results on algebraic tori. We refer to [5], [6] for the details. Let T be an algebraic torus defined over a field k. We denote $\hat{T}$ by $\mathbb{Z}$-module Hom(T, G_m) of rational characters of T. An extension K of k is called a splitting field of T if $\hat{T} = (\hat{T})_K$, where $(\hat{T})_K$ denotes the

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