ON FRAMED BORDISM

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1. INTRODUCTION

Let S^n denote the unit n-sphere with its standard differentiable structure in (n + 1)-dimensional Euclidean space. In [6], Novikov showed that if (m, n - m) ≠ (1, 1), (3, 3), 7, 7), then any framing of the stable tangent bundle of the product of spheres $S^m \times S^{n-m}$ determines, by the Pontrjagin-Thom construction, an element in the image of the stable J-homomorphism, $J: \pi_n(SO_k) \to \pi_{n+k}(S^k)$ for $k > n+1$. Our purpose here is to give a simple geometric proof of a generalization of Novikov's result; this generalization is stated in Theorem 1. We shall also obtain a result for the exceptional dimensions (m, n - m) = (1, 1), (3, 3), and (7, 7). For example, we shall see that any framing of any connected sum of 14-dimensional products of standard spheres determines either 0 or Toda's element σ^2 in the stable group $\pi_{14+k}(S^k)$. It is known that the parallelization of S^7 gives rise to a framing of $S^7 \times S^7$ that determines σ^2. For information about representing non-trivial elements in the homotopy groups of spheres by framings of Lie groups, we refer the reader to Atiyah, Smith [1], [7], Gershenson [4], Steer [8], and Wood [10].

In this paper all differentiable manifolds, with or without boundary, are compact, oriented, and of class C^∞. We denote the connected sum of r products of spheres of positive dimension by T_r^n; thus $T_r^n = (S^{m_1} \times S^{n-m_1}) \# \cdots \# (S^{m_r} \times S^{n-m_r})$, where # denotes the operation of connected sum. We shall prove the following theorem.

THEOREM 1. *Let* N^n *be a connected, differentiable* n-*manifold without boundary, and let* f *be a framing of the stable tangent bundle of the connected sum* $T_r^n \# N^n$. *If* n = 2, 6, 14 *and the integral homology group* $H_{n/2}(T_r^n)$ *is not zero, then make the following assumptions: for* n = 6, 14 *assume* $H_{n/2}(N^n) = 0$ *and* $(T_r^n \# N^n, f)$ *has Kervaire invariant zero; for* n = 2, *if* $H_1(N^2) = 0$, *assume* $(T_r^2 \# N^2, f)$ *has Kervaire invariant zero.*

Then there exists a framing g *of the stable tangent bundle of* N^n *such that* $(T_r^n \# N^n, f)$ *and* (N^n, g) *are framed-cobordant. Furthermore, if* n ≠ 2 *then* g *may be chosen such that the restrictions* $f \mid N^n$-disk *and* $g \mid N^n$-disk *are equal; hence* $f \mid T_r^n$-disk *extends to a framing* f_1 *of* T_r^n.

To see the need for the assumptions made in dimensions 2, 6, and 14 in this theorem, notice that if N^n is a homotopy sphere and g is any framing of its stable tangent bundle, then (N^n, g) has Kervaire invariant zero. Inasmuch as framed-cobordant manifolds have the same Kervaire invariant, it follows that (N^n, g) cannot be framed-cobordant to a framed manifold $(T_r^n \# N^n, f)$ with Kervaire invariant 1. In each of these dimensions, n = 2m = 2, 6, and 14, there is a framing f of

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