ON VANISHING EICHLER PERIODS AND CARLESON SETS

Thomas A. Metzger

1. INTRODUCTION

Let Γ be a Fuchsian group acting on the unit disk D in the complex plane, and let q be an integer, $q \geq 2$. An analytic function f defined on D is said to be an *automorphic form* of weight q with respect to Γ if $(f \circ \gamma)\gamma'^q = f$ for all γ in Γ.

The *Bers spaces* $A_q^p(\Gamma)$, $1 \leq p \leq \infty$, are defined as those Banach spaces of analytic automorphic forms of weight q such that

$$\|f\|_q^p = \iint_{D/\Gamma} |f(z)|^p (1 - |z|^2)^{pq-2}\, dx\, dy < \infty, \qquad 1 \leq p < \infty;$$

$$\|f\|_\infty = \sup_D |f(z)|(1 - |z|^2)^q < \infty, \qquad p = \infty.$$

Any analytic automorphic form f of weight q can be integrated $(2q - 1)$ times to get an analytic function $h \equiv I^{2q-1} f$ which satisfies

$$(h \circ \gamma)\gamma'^{1-q} = h + c(\gamma, f) \qquad \text{for all } \gamma \text{ in } \Gamma.$$

This $c(\gamma, f)$ is a polynomial of degree $s \leq 2q - 2$ and is called the *Eichler period* of f along γ. Bers [2] proved

THEOREM A. *If Γ is a group of the first kind, and the Eichler period of ϕ in $A_q^\infty(\Gamma)$ vanishes for all γ in Γ, then $\phi \equiv 0$.*

We shall extend this to say that if there exists a ϕ in $A_q^p(\Gamma)$ with vanishing Eichler period for all γ in Γ, then either $\phi \equiv 0$ or the limit set L is sparse in a special sense; *i.e.*, L is a Carleson set.

Conversely, Pommerenke [10] has recently shown that if L is a Carleson set, then there exists an f_0 in $A_2^\infty(\Gamma)$ such that $c(\gamma, f_0) = 0$ for all γ in Γ. I wish to thank Professor Pommerenke for our discussions on this topic. Also, I wish to thank the referee for pointing out a gap in the original proof of Theorem 1.

2. PRELIMINARIES

A closed set E of Lebesgue measure zero contained in ∂D is said to be a *Carleson set* if in the canonical representation of its complement $\partial D \backslash E$ as a countable union of disjoint open intervals I_n, the lengths $\ell(I_n)$ satisfy

Received January 28, 1976. Revisions received April 20, 1976, August 20, 1976, February 23, 1977, and April 24, 1977.

Michigan Math. J. 24 (1977).