GROWTH CONDITIONS AND UNIQUENESS FOR WALSH SERIES

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Let Ψ_0, Ψ_1, $\cdots$ denote the Walsh functions defined on the group 2^ω (see [2]). Let F be a finite-valued function belonging to $L^1(2^\omega)$, and let E^* be a countable subset of the group 2^ω. Skvortsov [5] has shown that if $S = \sum a_k \Psi_k$ is a Walsh series satisfying

$$\text{(1)} \qquad \liminf_{n \to \infty} S_{2^n}(x) \leq F(x) \leq \limsup_{n \to \infty} S_{2^n}(x), \quad x \notin E^*,$$

and if a_k converges to zero as $k \to \infty$, then S is the Walsh-Fourier series of F. Crittenden and Shapiro [1] have studied Walsh series which satisfy the weaker growth condition $\lim_{n \to \infty} 2^{-n} S_{2^n}(x) = 0$, $x \in 2^\omega$. Under this condition, they showed that if the right-hand side of (1) holds, and if

$$\text{(2)} \qquad \limsup_{n \to \infty} |S_{2^n}(x)| < \infty, \quad x \notin E^*,$$

then S is a Walsh-Fourier series.

We shall obtain the following result concerning uniqueness under an even weaker growth condition.

THEOREM. *Let* $S = \sum a_k \Psi_k$ *be a Walsh series and suppose that* (1) *holds for a finite-valued integrable function* F *and a countable subset* E^* *of the group* 2^ω. *Suppose further that*

$$\text{(3)} \qquad \liminf_{n \to \infty} 2^{-n} S_{2^n}(x) \leq 0 \leq \limsup_{n \to \infty} 2^{-n} S_{2^n}(x), \quad x \in E^* \cup D^*,$$

where D^* *is the set of points in the group* 2^ω *which terminate in* 0*'s or terminate in* 1*'s. Then* S *is the Walsh-Fourier series of* F.

Our result generalizes Skvortsov's theorem, but it does not generalize Crittenden and Shapiro's theorem. Instead, it restores the left-hand side of (1) in order to get rid of hypothesis (2). In connection with this, it is interesting to note that Šaginjan [4] has shown that condition (1) holds off a set of measure zero whenever (2) is satisfied. Hence, our theorem shows that if we insist that this null-set be countable, we can discard (2) altogether. For other related results, see [6] and [7].

To prove the theorem, for each $x \in [0, 1)$ set $f(x) = F(\mu(x))$, $\psi(x) = \Psi(\mu(x))$, and $s(x) = S(\mu(x))$, where μ is Fine's map from $[0, 1)$ to the group 2^ω (see [2]). Extend f and s to $(-\infty, \infty)$ as periodic functions of period 1. Let $E = \mu^{-1} E^*$, and let D be the set of dyadic rationals.

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