ON A QUESTION OF OLSEN CONCERNING COMPACT PERTURBATIONS OF OPERATORS

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1. INTRODUCTION

Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex separable infinite-dimensional Hilbert space $\mathscr{H}$, and $\mathscr{C}(\mathscr{H})$ the algebra of all compact linear operators on $\mathscr{H}$. The *essential norm* of T in $\mathscr{B}(\mathscr{H})$ is defined to be $\|T\|_e = \inf\{\|T+K\|: K \in \mathscr{C}(\mathscr{H})\}$. The main purpose of this paper is to answer a question posed by C. L. Olsen in her talk at the meeting of the American Mathematical Society at Washington, D. C. in January 1975. We show that *for each* $T \in \mathscr{B}(\mathscr{H})$ *there exists a* $K \in \mathscr{C}(\mathscr{H})$ *such that*

$$\|T + \lambda + K\| = \|T + \lambda\|_e \textit{ for every complex number } \lambda.$$

A few words seem appropriate to motivate the consideration of this problem. Recently much interest has been centered about the following unsolved problems.

(a) *Given* $T \in \mathscr{B}(\mathscr{H})$, *does there exist a* $K \in \mathscr{C}(\mathscr{H})$ *such that for any complex polynomial* p, $\|p(T+K)\| = \|p(T)\|_e$? An affirmative answer to (a) would imply the following results of Olsen and West.

1. (Olsen [8]) *Let* $\nu(T)$ *be a coset in the Calkin algebra. For a Hilbert space operator* T, *if* $p(\nu(T)) = 0$ *for some polynomial* p, *then there is a* $K \in \mathscr{C}(\mathscr{H})$ *with* $p(T+K) = 0$.

2. (West [13]) *If* $\lim_n \|\nu(T)^n\|^{1/n} = 0$, *then there is a* $K \in \mathscr{C}(\mathscr{H})$ *such that* $\lim_n \|(T+K)^n\|^{1/n} = 0$.

An affirmative solution to (a) would also answer a question raised by Arveson [2]: *If* $\nu(T)$ *is quasialgebraic, must there exist a* $K \in \mathscr{C}(\mathscr{H})$ *such that*

$$\|p_n(T+K)\|^{1/d(n)} \to 0\,?$$

(An element T of a Banach algebra is *quasialgebraic* if there is a sequence $\{p_n\}$ of monic polynomials of degree d(n) such that $\lim_n \|p_n(T)\|^{1/d(n)} = 0$.)

Question (a) seems to be very interesting, and the particular results of Olsen and West are nontrivial. However, it is far from being settled. For subnormal and essentially normal operators, positive results have been obtained in [10]. Even if we allow the compact operator to depend on the given polynomial, this problem is still open; that is,

(b) *Given* $T \in \mathscr{B}(\mathscr{H})$ *and a polynomial* p, *does there exist a* $K \in \mathscr{C}(\mathscr{H})$ *with* $\|p(T+K)\| = \|p(T)\|_e$?

Results of Olsen [9] along this line are:

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