BOUNDEDNESS AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A VOLTERRA EQUATION

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1. INTRODUCTION

We investigate the boundedness and the asymptotic behavior of the solutions of the nonlinear Volterra integral equation

$$(1.1) \qquad x'(t) + \int_{[0,t]} g(x(t-s))\, d\mu(s) = f(t) \qquad (t \in R^+)$$

(we write R^+ for the interval $[0, \infty)$). Here, g and f are given real functions, and μ is a given real (Radon) measure. A solution x is required to be locally absolutely continuous and satisfy (1.1) a.e. on R^+.

We suppose throughout that

$$(1.2) \qquad \mu \textit{ is positive definite,}$$

$$(1.3) \qquad g \in C(R), \quad \inf_{\xi \in R} G(\xi) > -\infty, \quad \textit{where } G(\xi) = \int_0^{\xi} g(\eta)\, d\eta.$$

If, in addition, one has

$$(1.4) \qquad f \in L^1(R^+),$$

$$(1.5) \qquad \limsup_{|x| \to \infty} |g(x)|\, (1 + |G(x)|)^{-1} < \infty,$$

then it is easy to show that every solution x of (1.1) satisfies

$$(1.6) \qquad \sup_{T \in R^+} G(x(T)) + Q(g \circ x, T, \mu) < \infty,$$

where $Q(\phi, T, \mu) = \int_0^T \phi(t) \int_{[0,t]} \phi(t-s)\, d\mu(s)\, dt \quad (\phi \in C[0, T])$.

The proof of this claim proceeds as follows: one multiplies (1.1) by $g(x(t))$ and integrates over $[0, T]$, getting

$$(1.7) \qquad G(x(T)) + Q(g \circ x, T, \mu) = G(x(0)) + \int_0^T g(x(t))\, f(t)\, dt.$$

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