STRUCTURE OF CERTAIN POLYNOMIAL HULLS

H. Alexander

1. INTRODUCTION

Let X be a compact subset of $\mathbb{C}^n$ with $\hat{X}$ its polynomially convex hull. Conditions on X are known which ensure that $\hat{X}\setminus X$ is a (possibly empty) complex analytic subvariety of $\mathbb{C}^n\setminus X$ of pure dimension one. The case for X being a smooth curve was treated by Stolzenberg [11] and for X a connected set of finite linear measure, by the author [1]. The basic ideas in the subject were introduced by Wermer [12] and Bishop [4]. Whenever the set $\hat{X}\setminus X$ is an analytic set of pure dimension one, it has locally finite "area". This is because pure k-dimensional subvarieties of $\mathbb{C}^n$ have locally finite $\mathscr{H}^{2k}$ measure [10], where $\mathscr{H}^s$ is s-dimensional Hausdorff measure. Our first result is a converse. It is also contained in the work of N. Sibony ([9], Theorem 17, p. 158). We have included a proof as a steppingstone for the generalization in Theorem 2 below.

THEOREM 1. *Let* X *be a compact subset of* $\mathbb{C}^n$. *If* $\mathscr{H}^2(\hat{X}\setminus X) < \infty$, *then* $\hat{X}\setminus X$ *is an analytic subvariety of* $\mathbb{C}^n\setminus X$ *of dimension one. More generally, if a point of* $\hat{X}\setminus X$ *has a neighborhood of finite* $\mathscr{H}^2$ *measure, then* $\hat{X}$ *is locally a pure one-dimensional variety in a neighborhood of the point.*

When the hypothesis of this theorem fails, it may occur that $\hat{X}\setminus X$ can be a countable union of varieties, without being a subvariety of $\mathbb{C}^n\setminus X$. For example, let A be a countable compact subset of $\mathbb{C}$ which is *not* discrete in its relative topology. Put $X = T\times A \subseteq \mathbb{C}^2$, where T is the unit circle. Then $\hat{X}\setminus X = U\times A$ (where U is the open unit disc) fails locally to be a variety at each point (z, α) with $|z| < 1$ for which α is a cluster point of A. Notice that the set $\hat{X}\setminus X$ has σ-finite $\mathscr{H}^2$ measure. (We will take this to mean that $\hat{X}\setminus X$ is a countable union of compact subsets each of which is of finite $\mathscr{H}^2$ measure.) The next result describes the structure of a hull with σ-finite $\mathscr{H}^2$ measure.

THEOREM 2. *Let* X *be a compact subset of* $\mathbb{C}^n$ *such that* $\hat{X}\setminus X$ *has* σ*-finite* $\mathscr{H}^2$ *measure. Then there exist a countable ordinal* μ *(possibly an integer) and a family of compact sets* K_α *with* $X\subseteq K_\alpha \subseteq \hat{X}$, *defined for each ordinal* $1\le\alpha\le\mu$, *such that*

(i) $K_1 = \hat{X}$ *and* $K_\mu = X$;

(ii) $K_\alpha \supsetneq K_\beta$ *for* $1\le\alpha<\beta\le\mu$;

(iii) *The set* $W_\alpha \equiv K_\alpha\setminus K_{\alpha+1}$ *is a relatively open dense subset of* $K_\alpha\setminus X$ *for* $1\le\alpha<\mu$;

(iv) W_α *is a nonempty pure one-dimensional analytic subvariety of* $\mathbb{C}^n\setminus K_{\alpha+1}$ *for* $1\le\alpha<\mu$;

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