# GROWTH OF NUMERICAL RANGES OF POWERS OF HILBERT SPACE OPERATORS

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## 1. INTRODUCTION

Let $\mathscr{H}$ denote a complex Hilbert space with inner product $(\cdot\,,\cdot)$, and let $\mathscr{B}(\mathscr{H})$ be the set of all bounded linear operators on $\mathscr{H}$. For $T \in \mathscr{B}(\mathscr{H})$, let $\sigma(T)$ denote the spectrum of T, and let W(T) denote the numerical range of T, $W(T) = \{(Tx, x): x \in \mathscr{H}, \|x\| = 1\}$.

The power inequality $|W(T^n)| \leq |W(T)|^n$ for numerical radii was first proved by C. A. Berger ([1], [4, Problem 176]). This inequality gives an estimate of the maximum rate of growth of the numerical ranges of the powers of an operator. In this paper, we study the minimum rate of growth of the numerical ranges of the powers of non-Hermitian operators. Using von Neumann's theory of spectral sets, we prove that *if* $\sigma(T) \subset (\gamma, \infty)$ *with* $\gamma > 0$, *and if* T *is not Hermitian, then there is a positive integer* $n_0$ *such that* $\{z \in \mathbb{C}: |z| \leq \gamma^n\} \subset W(T^n)$ *whenever* $n \geq n_0$. In particular, if $\sigma(T) \subset (1, \infty)$, then either T is Hermitian, or for each bounded set $\Omega$ of complex numbers, there exists an integer $n_0(\Omega)$ such that $\Omega \subset W(T^n)$ whenever $n \geq n_0(\Omega)$. In the last part of this paper, we show that $\Re(T^n x, x) \geq 0$ for all $x \in \mathscr{H}$ and for $n = 1, 2, \cdots, k$, if and only if the closed sector

$$\{z \in \mathbb{C}: |\operatorname{Arg} z| \leq \pi/2k\} \cup \{0\}$$

is spectral for T; moreover, $\|\Im\, Tx\| \leq \tan(\pi/2k)\, \|\Re\, Tx\|$ for all $x \in \mathscr{H}$.

## 2. NOTATION AND PRELIMINARIES

Let $\mathbb{C}$ denote the set of complex numbers, and $\mathbb{R}$ the set of real numbers. For $\Omega \subset \mathbb{C}$, we denote by $Co(\Omega)$ the convex hull, by $Cl(\Omega)$ the closure, by $Int(\Omega)$ the interior of $\Omega$, and by $\Re(\Omega)$ the set $\{(z + \bar{z})/2: z \in \Omega\}$. For $\gamma \in \mathbb{R}$, we write $\Omega \geq \gamma$ (or $\Omega > \gamma$) if $\Omega \subset \mathbb{R}$ and each number $\lambda$ in $\Omega$ satisfies the condition $\lambda \geq \gamma$ (or $\lambda > \gamma$). Let $\Delta(r) = \{z \in \mathbb{C}: |z| \leq r\}$. Let $\Sigma(\phi)$ denote the closed sector of the complex plane symmetric with respect to the real axis, with vertex at the origin, and with angular opening $2\phi$; and note that $\Sigma(\pi/2)$ denotes the right half-plane.

For $T \in \mathscr{B}(\mathscr{H})$, $\Re T = (T + T^*)/2$ and $\Im T = (T - T^*)/2i$. We say T is *positive* and write $T \geq 0$ if $W(T) \geq 0$; if T is also invertible, we write $T > 0$. By $T_1 \geq (>)\, T_2$, we mean $(T_1 - T_2) \geq (>)\, 0$. We say that T is *accretive* if $\Re T \geq 0$, *strictly accretive* if $\Re T > 0$.

Let $T \in \mathscr{B}(\mathscr{H})$, and let $\Omega$ be an open set of complex numbers containing $\sigma(T)$, whose boundary $\partial\Omega$ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense. If f is a function analytic on some neighborhood of $Cl(\Omega)$, then we define f(T) by the Riesz-Dunford integral [3, Chapter VII]

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