SPECTRA OF OPERATOR EQUATIONS

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Halpern [2] and Akemann and Ostrand [1] have recently characterized the spectra of operator equations on W*- and C*-algebras in terms of the spectra of elements in suitable homomorphic images of the algebra in question. The purpose of this note is to show that in certain cases of special interest (*e.g.*, derivations and inner automorphisms), the spectrum of the operator equation on a W*-algebra can be described in an "internal" fashion.

If a is an $n \times n$ complex matrix and δ denotes the inner derivation induced by a on the algebra of all $n \times n$ complex matrices, it is well known that

$$\operatorname{sp}\delta = \{\alpha - \beta \mid \alpha,\ \beta \in \operatorname{sp} a\}.$$

(Indeed, the same proof applies if the matrix algebra is replaced by any prime algebra over a field F and a is any algebraic element of the algebra such that the minimum polynomial of a splits into linear factors in F.) In a more general setting, this statement may not be true (*e.g.*, [1, Example 2] or, more trivially, a commutative algebra). However, the same derivation may be induced by many different elements; in particular, if z is central, a and $a + z$ induce the same derivation. Corollary 1 says that on a W*-algebra, $\operatorname{sp}\delta$ is the intersection over all a which induce δ of $\{\alpha - \beta \mid \alpha,\ \beta \in \operatorname{sp} a\}$.

Let A be a W*-algebra with center Z. Let $a, b \in A$ and let ϕ_i, ψ_i be holomorphic functions on domains containing $\operatorname{sp} a$ and $\operatorname{sp} b$, respectively. Define the operator T in A by

$$Tx = \sum \phi_i(a)\, x\, \psi_i(b).$$

The following lemma is easily extracted from [2, Proposition 6] and its proof.

LEMMA. *Given $\gamma \notin \operatorname{sp} T$, there exists a finite set of mutually orthogonal projections $e_1, \cdots, e_n \in Z$ with $\sum e_j = 1$, such that γ is not in*

$$\Big\{\sum_i \phi_i(\alpha)\,\psi_i(\beta) \;\Big|\; \alpha \in \operatorname{sp}_{Ae_j} ae_j,\ \beta \in \operatorname{sp}_{Ae_j} be_j\Big\}$$

for any $j = 1, \cdots, n$.

THEOREM. (i) *If* $Tx = ax + xb$, *then*

$$\operatorname{sp} T = \bigcap \{\alpha + \beta \mid \alpha \in \operatorname{sp}(a + z),\ \beta \in \operatorname{sp}(b - z)\},$$

where the intersection is taken over all $z \in Z$.

(ii) *If* $Tx = axb$, *then*

Received November 13, 1975. Revision received January 19, 1976.
Research supported in part by a grant from the National Science Foundation.

Michigan Math. J. 23 (1976).