A LOCAL FORM OF LAPPAN'S FIVE-POINT THEOREM FOR NORMAL FUNCTIONS

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A necessary and sufficient condition for a meromorphic function f to be normal in the unit disc D is that

$$\sup_{z \in D} f^{\#}(z)(1 - |z|^2) < \infty, \quad \text{where } f^{\#}(z) = \frac{|f'(z)|}{1 + |f(z)|^2}$$

(see [3]). Ch. Pommerenke [5] asked whether there exists a class of sets $\mathscr{E}$ such that if there is a set $E \in \mathscr{E}$ for which the quantity $\sup_{z \in f^{-1}(E)} f^{\#}(z)(1 - |z|^2)$ is sufficiently small then f is a normal function. P. Lappan's beautiful answer [2, Theorem 2] shows that f is a normal function in D if and only if for some five-point subset A of the extended plane Ω, $\sup_{z \in f^{-1}(A)} f^{\#}(z)(1 - |z|^2) < \infty$. This result has an elegant proof which combines a result of A. J. Lohwater and Ch. Pommerenke [4, Theorem 1] with the Nevanlinna theory on completely ramified values of a function meromorphic in the finite plane W. Our contribution is to make precise and to generalize slightly the result of Lohwater and Pommerenke, and this in turn leads to an extended form of Lappan's theorem.

Before proving the extended Lohwater-Pommerenke theorem we attend to a few details. Let ρ be the hyperbolic distance on D. For $a \in D$ and $r > 0$, let $D(a, r) = \{z \in D: |z - a| < r\}$ and $N(a, r) = \{z \in D: \rho(a, z) < r\}$. Note that if $\{z_n\}$ and $\{z_n'\}$ are two sequences in D such that $\rho(z_n, z_n') \to 0$ $(n \to \infty)$, then

(0) $$(1 - |z_n|)/(1 - |z_n'|) \to 1 \qquad (n \to \infty).$$

In the sequel we use the Hardy-Littlewood "o" notation.

THEOREM 1. *Let* f *be meromorphic in* D *and* $\alpha \geq 1$. *The following two statements are equivalent.*

(i) *There exists a sequence* $\{z_n\}$ *such that*

$$\lim_{n \to \infty} f^{\#}(z_n)(1 - |z_n|^2)^{\alpha} = \infty.$$

(ii) *There exist a sequence* $\{\zeta_n\}$ *and a sequence of positive numbers* $\{\varepsilon_n\}$ *satisfying*

(1) $$\varepsilon_n = o(1 - |\zeta_n|^2)^{\alpha} \qquad (n \to \infty),$$

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