A NOTE ON DIRECT INTEGRALS OF SPECTRAL OPERATORS

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1. INTRODUCTION

It is easy to see that a decomposable operator is normal if and only if almost all of its direct integrands are normal. The following theorem, stated by T. R. Chow in [1], provides a corresponding characterization of decomposable spectral operators.

THEOREM 1.1. *Let* A *be a bounded decomposable operator on the separable Hilbert space* H,

$$A = \int_{\Sigma}^{\oplus} A(s)\, d\mu(s)\,.$$

Then A *is spectral if and only if*

(i) $A(s)$ *is spectral for* μ*-almost every* s, *with spectral measure* E_s,

(ii) *for each Borel set* $B \subseteq \mathbb{C}$, *the function* $S \mapsto E_s(B)$ *is measurable on* Σ,

(iii) $\sup\{\mu\text{ - ess sup}\, \|E_s(B)\|;\ B \text{ a Borel set in } \mathbb{C}\} < \infty$,

(iv) $\lim(\mu\text{ - ess sup}\, \|N(s)^n\|^{1/n}) = 0$, *where* $N(s)$ *is the radical part of* $A(s)$.

A gap in Chow's proof was filled by M. J. J. Lennon in [6]. Lennon also gave examples to show that the theorem would no longer be true if either of the conditions (iii) or (iv) were omitted. On the other hand, both Chow and Lennon conjectured that condition (ii) is redundant. In Section 2 of this note, we establish this conjecture, essentially by reducing the problem to the (known) case of normal operators. As a by-product of this technique, we show in Section 3 that every direct integral of spectral operators is in fact a direct sum of spectral operators.

We use the notation established in [6]. In particular, all operators discussed will be bounded operators acting on separable Hilbert spaces, and we follow Dixmier's formulation of direct-integral theory [2]. A complete discussion of spectral operators can be found in [3]. Beyond the basic definitions, the main fact needed below is the canonical decomposition of spectral operators; this material can be found in the first four sections of [3].

2. REDUNDANCY OF CONDITION (ii)

The main result to be proved in this section is the following theorem.

THEOREM 2.1. *Let* $s \mapsto A(s)$ *be a measurable field of spectral operators, and write* E_s *for the spectral measure of* $A(s)$. *Then for each Borel set* B *the field* $s \mapsto E_s(B)$ *is measurable.*

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