ON CONJUGACY CLASSES IN THE TEICHMÜLLER MODULAR GROUP

Jane Gilman

Let S be a compact Riemann surface of genus g with n punctures, and let M(g, n) be the Teichmüller modular group of S. Let $\lambda(p, g, n)$ denote the number of conjugacy classes of elements of prime order p in M(g, n). The purpose of this paper is to obtain an explicit formula for $\lambda(p, g, n)$ when $g \geq 2$. (W. J. Harvey has considered this problem [1]. He obtained a generating function for $\lambda(p, g, 0)$ in [1], but he has recently pointed out that his function actually gives the number of conjugacy classes of subgroups of M(g, 0) of order p and not $\lambda(p, g, 0)$.)

Let $|M_{T,p}^x|$ be the number of distinct (p - 1)-tuples of nonnegative integers $(n_1, \cdots, n_{p-1})$ with

$$\sum_{i=1}^{p-1} n_i = T \quad \text{and} \quad \sum_{i=1}^{p-1} i n_i \equiv x \pmod{p},$$

where T is a fixed integer. In Section 1 we obtain a set of invariants for each conjugacy class; these invariants make it clear that to compute $\lambda(p, g, n)$ we need to know $|M_{T,p}^x|$. In Section 2 we compute $|M_{T,p}^x|$, and in Section 3 we compute $\lambda(p, g, n)$.

1. INVARIANTS FOR A CONJUGACY CLASS

Let S and S' be compact surfaces of genus g with n punctures, so that $S = \bar{S} - Q$ and $S' = \bar{S}' - Q'$, where Q and Q' are sets of n points on the compact surfaces $\bar{S}$ and $\bar{S}'$. Let h and h' be homeomorphisms of S and S', respectively, whose pth powers are homotopic to the identity.

Definition. The pairs (S, h) and (S', h') are *topologically equivalent* if there is a homeomorphism f of S onto S' with $fhf^{-1} \simeq h'$, where $\simeq$ denotes homotopy (see [3]). This is clearly an equivalence relation.

LEMMA 1. (i) *The number of conjugacy classes of elements of order* p *in* M(g, n) *is equal to the number of topological equivalence classes of pairs* (S, h), *where* S *is of genus* g *with* n *punctures and* $h^p \simeq$ identity.

(ii) *For each pair* (S, h) *there is an equivalent pair* (S', h') *in which* h' *is conformal.*

Proof. Fix a surface S. Given any pair (S', h'), let f be a homeomorphism of S' onto S. Then (S', h') is topologically equivalent to $(S, fh'f^{-1})$. Thus, to count topological equivalence classes we need only consider pairs where the surface is fixed. But (S, h) is equivalent to (S, h') if and only if h and h' are conjugate in M(g, n).

Received December 4, 1974.

Michigan Math. J. 23 (1976).