D + M CONSTRUCTIONS WITH GENERAL OVERRINGS

J. W. Brewer and E. A. Rutter

Suppose that T is a domain and K is a field that is a retract of T, that is, suppose T = K + M, where M is a maximal ideal of T. Each subring D of K determines a subring R = D + M of T. This construction has been studied extensively in two situations. The first systematic investigation of the properties of R is due to R. Gilmer [8, Appendix 2, p. 558] and Gilmer and W. Heinzer [9], who required that T be a valuation domain. More recently, a similar investigation has been conducted under the hypothesis that T = K[X], M = XK[X], and K is the quotient field of D [4]. The interest in this case arises because R is the symmetric algebra of the D-module K. In both cases the properties of R are related to those of D; in the case of a valuation domain, the relationship of D to K also plays an important role. In this paper, we investigate the construction described above, without placing any limitations on T. The authors find it remarkable that things proceed as well as in the special cases considered earlier. Of course, in the more general context the properties of T and M, or more often of T_M, also play a crucial role.

More specifically, we focus attention on four properties: we obtain necessary and sufficient conditions for R to be a coherent domain, a Prüfer domain, a Noetherian domain, and a GCD-domain. What is most satisfying is that the conditions are expressed solely in terms of the properties of the components of the construction. If K is the quotient field of D, it is also possible to describe the prime-ideal lattice of R and thus to compute the Krull dimension of R. If R is a Prüfer domain, so are D and T. Their ideal class groups are shown to be related by a short exact sequence. This yields conditions for R to be a Bézout domain. Unfortunately, if R is a Prüfer domain it has the n-generator property whenever D and T do. Thus, this construction casts no light on whether invertible ideals in Prüfer domains can require more than two generators. The paper concludes with a brief consideration of methods for obtaining domains T of the form K + M that satisfy the conditions of the theorems.

It is undoubtedly possible to characterize other properties. We have limited ourselves to these four because they have received attention in the special contexts investigated earlier, and because they seem adequate to demonstrate that such problems can often be handled in more generality than had previously seemed feasible. It seems quite likely that at least some of the results of this paper can be extended to a somewhat more general situation. As Gilmer [8] noted, the assumption that K is a retract of T is often not essential. Instead, it can be assumed that K = T/M, in which case R is replaced by the pullback of T and D. However, we have chosen to follow Gilmer's lead in this regard, and for the sake of clarity and simplicity we limit ourselves to the case of a retract.

Our interest in this problem was kindled by the recent paper of D. Dobbs and I. Papick [5], which gives necessary and sufficient conditions for R to be coherent when T is a valuation domain. We have benefited from possessing a preprint of that paper. We have also been helped by access to Doug Costa's thesis, which contains results concerning the case where T = K[X], and by correspondence with him concerning this problem. Finally, we must thank our colleagues Ray Heitmann and

Received September 12, 1975.

Michigan Math. J. 23 (1976).