THE CURVATURE OF $\alpha\mathrm{I} + \beta\mathrm{II} + \gamma\mathrm{III}$ ON A SURFACE IN A 3-MANIFOLD OF CONSTANT CURVATURE

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1. INTRODUCTION

There has been growing interest during recent years in the differential-geometric properties of nonstandard metrics on immersed surfaces. This interest has centered on the study of the geometry of the second and third fundamental forms II and III on compact surfaces in Euclidean 3-space E^3. (See [10] for an extensive review, or [12], [17], [18], and [21].) However, in [3] and [4], N. V. Efimov obtained impressive results about open complete surfaces in E^3 by studying the properties of the metric $|K|\,\mathrm{I}$, where I is the first fundamental form and K is Gauss curvature.

In this paper, we consider a surface immersed in a Riemannian 3-manifold of constant curvature, and for arbitrary constants α, β, and γ, we compute the curvature of the (not necessarily Riemannian) metric $\Lambda = \alpha\mathrm{I} + \beta\mathrm{II} + \gamma\mathrm{III}$ wherever Λ is nondegenerate. Our formula extends work due to N. Hicks [5] and J. A. Wolf [23], and it yields as a minor byproduct the fact that the curvature K(III) on such a surface is just the ratio of intrinsic to extrinsic curvature. It is remarkable that this simple formula (which can easily be verified directly) seems to have first appeared in the literature just recently as a special case of a more general result due to B. Wegner [22].

The applications included in this paper are fairly pedestrian. Perhaps others will find more significant uses for our formulas. But we hope this article will encourage the study of metrics other than I, II, and III that are nonetheless determined by the immersion of a surface in some Riemannian 3-manifold. (See [13], [14], and [15].) The goal of such efforts should be the accumulation of information useful in solving problems in-the-large.

2. THE BASIC COMPUTATION

Suppose $X\colon S \to \mathcal{M}$ is a C^3 immersion of a surface S in some Riemannian 3-manifold $\mathcal{M}$, and that ν is a unit normal vector field on the immersed surface. Let D denote the covariant differential in $\mathcal{M}$ [6, p. 56] and let $\cdot$ denote the inner product provided by the Riemannian metric on $\mathcal{M}$ [6, p. 21]. Then there are two restrictions [16, p. 527] on the fundamental forms

$$\mathrm{I} = DX \cdot DX = E\,dx^2 + 2F\,dx\,dy + G\,dy^2,$$

$$\mathrm{II} = -DX \cdot D\nu = L\,dx^2 + 2M\,dx\,dy + N\,dy^2$$

of the immersion. First, the intrinsic curvature K(I) [6, p. 29] of the official Riemannian metric I on S is related to the extrinsic curvature

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