REDUCTIVE OPERATORS THAT COMMUTE WITH A COMPACT OPERATOR

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A bounded operator T on a Hilbert space $\mathscr{H}$ is *reductive* if every invariant subspace of T reduces T. It is well known that every reductive operator is normal if and only if every operator has a nontrivial invariant subspace [4]. In 1963, T. Andô [1] showed that every compact reductive operator is normal, and in 1968 P. Rosenthal [10] was able to extend this result by showing that every polynomially compact reductive operator is normal. In this paper we use the work of V. I. Lomonosov [7] to generalize these results; the principal theorem is that a reductive operator that commutes with an injective compact operator must be normal.

Rosenthal [11] has recently shown that if an injective compact operator is contained in the commutant of a reductive algebra, then the reductive algebra must be self-adjoint. In addition, recent papers by E. Azoff [2] and A. I. Loginov and V. S. Šul'man [6] contain generalizations of Rosenthal's result. Rosenthal's theorem is stronger than our Theorem 1; however, the techniques used herein are quite different from Rosenthal's, and several of the intermediate results are of interest in themselves. The proof of the first proposition is essentially in [1] and [10]; we include it here for completeness.

PROPOSITION 1. *Let* C *be a nonzero compact operator. Let* $\mathscr{G}$ *be a family of subspaces with the following properties:*

(i) $\mathscr{G}$ *is totally ordered by reverse inclusion;*

(ii) *each subspace* $\mathscr{M}$ *in* $\mathscr{G}$ *reduces* C;

(iii) *for each* $\mathscr{M}$ *in* $\mathscr{G}$, $\|C \mid \mathscr{M}\| = \|C\|$.

Then the intersection $\mathscr{M}_0 = \bigcap \mathscr{G}$ *is nonzero and* $\|C \mid \mathscr{M}_0\| = \|C\|$.

Proof. For each $\mathscr{M} \in \mathscr{G}$, $C \mid \mathscr{M}$ is a compact operator, and since a compact operator achieves its norm, there is a unit vector $f_{\mathscr{M}} \in \mathscr{M}$ such that $\|Cf_{\mathscr{M}}\| = \|C \mid \mathscr{M}\| = \|C\|$. Because the $f_{\mathscr{M}}$ all lie in the unit ball of the Hilbert space and the unit ball is weakly compact, there is a weak cluster point f_0 of the set $\{f_{\mathscr{M}}\}$ in the unit ball. We consider $\{f_{\mathscr{M}}\}$ as a net, indexed by the totally ordered set $\mathscr{G}$; some subnet of $\{f_{\mathscr{M}}\}$ converges to f_0, and we assume without losing generality that the full net $\{f_{\mathscr{M}}\}$ converges to f_0. Since C is compact, $Cf_{\mathscr{M}} \to Cf_0$ in norm, whence $\|Cf_0\| = \|C\|$; because C is nonzero, f_0 is nonzero. Moreover, for each $\mathscr{M}'$ in $\mathscr{G}$, the tail of the net $\{f_{\mathscr{M}}\}$ lies in $\mathscr{M}'$ (since $\mathscr{G}$ is ordered by reverse inclusion), so that f_0 lies in $\mathscr{M}_0$. Thus $\mathscr{M}_0$ is nonzero and $\|C \mid \mathscr{M}_0\| = \|C\|$.

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