ON A DIFFERENTIAL-DIFFERENCE EQUATION

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In this paper we consider the entire and the meromorphic solutions $f(z)$ of the differential-difference equation

$$f(z+1) = \exp[P(z)]f'(z), \tag{1}$$

where

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n \qquad (a_0 \neq 0) \tag{2}$$

is an arbitrary polynomial of degree $n \geq 1$.

We agree to say that

(a) a meromorphic function $f(z)$ is *properly meromorphic* if $f(z)$ has at least one pole,

(b) an infinite set E of entire functions is *linearly independent* (over the field of complex numbers) if each finite system of functions of the set E is linearly independent.

We prove that the equation (1) has no properly meromorphic solutions but has a linearly independent infinite set E (with cardinality of the continuum) of entire solutions. Furthermore, for each $\rho \geq n+1$ in the case $n > 1$ and for $\rho > 2$ in the case $n = 1$, the equation (1) has a linearly independent set E_ρ (with cardinality of the continuum) of entire solutions of order ρ.

The equation (1) has no nontrivial entire solutions $f(z)$ $(f(z) \not\equiv 0)$, either of order $\rho < n+1$ or of minimal type with respect to the order $\rho = n+1$. On the other hand, this equation has entire solutions of normal type with respect to the order $\rho = n+1$.

Besides the equation (1), we shall also consider the inhomogeneous equation

$$f(z+1) = \exp[P(z)]f'(z) + g(z) \tag{3}$$

with an entire or meromorphic free term $g(z)$.

Remark 1. In the case $n = 0$, the equation (1) reduces to an equation with constant coefficients. Such equations have been at the center of wide research, and we do not include them in our study although the method used in this paper may be applied to some extent to these equations also.

Remark 2. The results announced above on the entire solutions of the equation (1) contain an answer to the following question of Hurwitz [2, p. 752]:

Is it possible for a power series

$$h(\xi) = \sum_{k=0}^{\infty} a_k(\xi - \xi_0)^k \qquad (h(\xi) \not\equiv c \exp \xi,\ c = \text{const})$$

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