INVERSE LIMITS AND THE COMPLETENESS OF QUOTIENT GROUPS

David Wigner

In [5], G. Köthe gives an example of a complete topological vector space and a closed subspace such that the quotient space is not complete. In this paper we consider the question under what conditions the quotient of a complete abelian topological group by a closed subgroup is complete. We give sufficient conditions on the closed subgroup, and in general we define functors L^i from abelian topological groups to abelian groups such that the vanishing of L^1 for the closed subgroup implies the completeness of the quotient. The L^i are shown to be closely related to the derived functors of the inverse limit, and we can conclude that the derived functors of the inverse limit of a strongly dense inverse system (in the sense of R. F. Arens [1]) of complete metrizable abelian groups depend only on the natural topology of the inverse limit. All topological groups considered in this paper will be abelian, but not necessarily Hausdorff. We declare a sequence $0 \to A \xrightarrow{\sigma} B \xrightarrow{\tau} C \to 0$ of abelian topological groups and continuous homomorphisms to be exact if it is exact as a sequence of abstract groups, σ is a homeomorphism onto its range, and τ is an open mapping. We use the following facts about exact sequences; the proofs are not difficult, and we omit them.

LEMMA 1. *Let* $0 \to A \to B \to C \to 0$ *be an exact sequence, and let* α: $A \to A'$ *and* γ: $C'' \to C$ *be continuous homomorphisms. Then there exist commutative diagrams*

(Q)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

and

(Q*)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\ & & {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

with exact rows.

Here B'' is the fiber product of C'' and B over C. This implies that the category of abelian topological groups is a quasi-abelian category in the sense of N. Yoneda [7].

LEMMA 2. *Let*

Received October 14, 1974.

Michigan Math. J. 22 (1975).