INTEGRAL DOMAINS THAT SATISFY GAUSS'S LEMMA

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INTRODUCTION

Let D be a commutative integral domain with a unity element, and let D[x] denote the ring of polynomials with coefficients in D. For a polynomial f(x) in D[x], the *content of* f, denoted by A_f, is defined to be the ideal of D generated by the coefficients of f. The polynomial f is said to be *primitive over* D in case A_f is contained in no proper principal ideal of D, or equivalently, if no nonunit of D divides every coefficient of f. Primitive polynomials arise in the classical theory of unique factorization domains (UFD's) and in the theory of GCD-domains--those domains in which every pair of elements has a greatest common divisor--in the proof that both the class of all GCD-domains and the subclass of all UFD's are closed under polynomial extensions. Specifically, they appear in the preliminary result that if D is a GCD-domain, then the product of two primitive polynomials over D is primitive. This proposition is usually called Gauss's Lemma.

In this paper, we investigate the class of domains with the property of satisfying the conclusions in Gauss's Lemma. This property, which for obvious reasons we call the GL-property, is defined formally as follows:

Definition. A domain D has the GL-*property* if the product of two primitive polynomials over D is always a primitive polynomial.

In a related study of primitive polynomials over an arbitrary domain, H. T. Tang [6] presents a new concept closely related to primitivity, by defining a polynomial f(x) in D[x] to be *superprimitive over* D in case $A_f^{-1} = D$. Tang shows that, without any restrictions on D, every superprimitive polynomial is primitive, and furthermore, that the product of a primitive polynomial and a superprimitive polynomial is again primitive [6, Theorems C and D, p. 374]. The latter result is a generalization of Gauss's Lemma, since over a GCD-domain, a polynomial is primitive if and only if it is superprimitive [6, Theorem H, p. 375]. These results lead naturally to the study of the following property:

Definition. A domain D is said to have the PSP-*property* if every primitive polynomial over D is superprimitive.

In Section 1 of this paper, we characterize both the GL-property and the PSP-property in ideal-theoretic terms, and we derive a number of properties of domains with the GL-property, including the fact that irreducible elements are necessarily prime. Section 2 is largely devoted to the relation between the PSP-property and the GL-property. Since it follows directly from Tang's results cited earlier that the PSP-property implies the GL-property, we ask whether the reverse implication is true in general. We present a counterexample to this implication, but we also present some conditions on a domain that together with the GL-property are sufficient to imply the PSP-property. Additional results in Section 2 include an embedding theorem for a domain with the GL-property in a domain with the PSP-property,

Received December 24, 1974.

Michigan Math. J. 22 (1975).