QUASIALGEBRAIC OPERATORS, COMPACT PERTURBATIONS, AND THE ESSENTIAL NORM

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1. INTRODUCTION AND NOTATION

Quasialgebraic operators generalize algebraic operators, in the same way that quasinilpotent operators generalize nilpotent operators. An element T of a Banach algebra is *quasialgebraic* if one can find a sequence $\{p_n\}$ of monic polynomials with $\deg p_n = d(n)$, such that $\lim_n \|p_n(T)\|^{1/d(n)} = 0$. This concept was first introduced by P. R. Halmos in [14], where he related it to the potential-theoretic notion of capacity. As a starting point for our paper, we rely on some of his observations and techniques to focus on this question: if T is a bounded operator on a Hilbert space, and the coset $\nu(T)$ in the Calkin algebra is quasialgebraic, does it contain a compact perturbation of T that is quasialgebraic with respect to the same sequence of polynomials?

More precisely, W. B. Arveson has asked: if $\{p_n\}$ is a sequence of monic polynomials of degrees $d(n)$ such that $\lim_n \|p_n(\nu(T))\|^{1/d(n)} = 0$, does there exist a compact K such that $\lim_n \|p_n(T+K)\|^{1/d(n)} = 0$? Like other questions involving the structure of the Calkin algebra, it is recalcitrant. But it deserves attention, for an affirmative answer would imply two previous results: for a Hilbert space operator T, C. L. Olsen has proved that if $p(\nu(T)) = 0$, for some polynomial p, then there is a compact K with $p(T+K) = 0$ [17]; T. T. West has shown that if $\lim_n \|[\nu(T)]^n\|^{1/n} = 0$, then there is a compact K such that $\lim_n \|(T+K)^n\|^{1/n} = 0$ [24, Theorem 7.5]. In other words, an algebraic coset in the Calkin algebra contains an algebraic operator, and a quasinilpotent coset contains a quasinilpotent operator.

If the answer to Arveson's question is yes, then a quasialgebraic coset in the Calkin algebra must contain a quasialgebraic operator. In fact, we show that even more is true, by observing that a coset and every element in it must have the same capacity [Section 2]. These results were also obtained independently by David S. G. Stirling [22]. Complications arise when we insist that some compact perturbation of T be quasialgebraic with respect to the *same sequence of polynomials* as $\nu(T)$. However, if the sequence $\{p_n\}$ of monic polynomials has a subsequence of bounded degree, then, using Halmos's techniques and Olsen's theorem, we can easily answer Arveson's question. In any case, an application of a theorem of J. G. Stampfli [21] enables us to answer a weakened version of the question [Section 3]; that is, if a Hilbert space operator T is such that $\lim_n \|p_n(\nu(T))\|^{1/d(n)} = 0$ for a sequence $\{p_n\}$ of monic polynomials with $\deg p_n = d(n)$, then there exist a compact K and a sequence $\{s(n)\}$ of positive integers such that

$$\lim_n \|[p_n(T+K)]^{s(n)}\|^{1/d(n)s(n)} = 0.$$

Received August 26, 1973.

C. L. Olsen's contribution to this research was partially supported by National Science Foundation Grant PO37621.

Michigan Math. J. 21 (1974).