UNIFORM ALGEBRAS CONTAINING THE REAL AND IMAGINARY PARTS OF THE IDENTITY FUNCTION

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A uniform algebra on $\Gamma = \{z: |z| = 1\}$ is a subalgebra of $C(\Gamma)$ that is closed under the topology of the supremum norm, contains the constants, and separates the points of Γ. The canonical example is the disk algebra A, which is the uniform algebra consisting of all functions in $C(\Gamma)$ that extend continuously to $\{z: |z| \leq 1\}$ to be analytic on $D = \{z: |z| < 1\}$. In a recent paper [4], J. M. F. O'Connell shows that if B is a uniform algebra with $\Re B = \Re A$, then there exists a homeomorphism Φ of Γ onto Γ such that

$$B = A \circ \Phi = \{f \circ \Phi: f \in A\}.$$

W. P. Novinger [3] generalizes this result to the setting in which it is only assumed that $\Re B \supseteq \Re A$. He shows that in this case either $B = C(\Gamma)$ or $B = A \circ \Phi$ for some homeomorphism Φ. We show that to obtain the latter conclusion, it is sufficient to assume that $\Re B$ contains the real and imaginary parts of the identity function Z.

THEOREM 1. *Let B be a uniform algebra on Γ such that $\Re B$ contains $\Re Z$ and $\Im Z$. Then either $B = C(\Gamma)$ or there exists a homeomorphism Φ of Γ onto Γ such that $B = A \circ \Phi$.*

Proof. By hypothesis, there exist functions ψ and ϕ in B such that $\Re\psi = \Re Z$ and $\Im\phi = \Im Z$.

Case 1. Either ψ or ϕ is one-to-one on Γ. We shall assume that ψ is one-to-one on Γ. The proof for the case where ϕ is one-to-one is similar. Let W denote the interior of the Jordan curve $\psi(\Gamma)$. Let f denote the Riemann mapping of W onto D; then f extends continuously to $\overline{W}$, mapping $\psi(\Gamma)$ homeomorphically onto Γ. By Mergelyan's theorem, f can be uniformly approximated by polynomials on $\psi(\Gamma)$, and thus $\Phi = f \circ \psi$ is in B. Hence, $A \circ \Phi \subseteq B$, or equivalently, $A \subseteq B \circ \Phi^{-1}$. Applying Wermer's maximality theorem to the uniform algebra $B \circ \Phi^{-1}$, we see that either $B \circ \Phi^{-1} = C(\Gamma)$ or $B \circ \Phi^{-1} = A$. It follows immediately that $B = C(\Gamma)$ or $B = A \circ \Phi$.

Before proceeding to Case 2, we shall establish some useful results.

LEMMA 1. *Let B be a uniform algebra on Γ containing functions ψ and ϕ with $\Re\psi = \Re Z$ and $\Im\phi = \Im Z$. If $\psi(z_1) = \psi(z_2)$ or $\phi(z_1) = \phi(z_2)$ and E_1 and E_2 are the two closed subarcs of Γ with end points z_1 and z_2, then E_1 and E_2 are peak sets for B. Furthermore, $B \mid E_j$ is a closed subalgebra of $C(E_j)$ for $j = 1, 2$.*

Proof. If $z_1 = z_2$, then the conclusion is trivial. We shall assume that $\psi(z_1) = \psi(z_2)$. If $\phi(z_1) = \phi(z_2)$, the proof is similar. Note that our assumption implies that $z_2 = \bar{z}_1$.

Let K be the union of $\psi(\Gamma)$ and the bounded components of $\mathbb{C} - \psi(\Gamma)$. There exists a closed rectangle R containing $\psi(E_2)$ such that one edge of R is contained in $\{z: \Re z = \Re\psi(z_1)\}$. Let f be the Riemann mapping of int R onto D; then f

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