COMPACT, TOTALLY DISCONNECTED SETS THAT CONTAIN K-SETS

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It is well known that every infinite subset of a discrete abelian group contains an infinite Sidon set. In this paper, we present two analogous theorems on K-sets in nondiscrete, locally compact abelian groups. Each theorem says, roughly, that certain compact, metrizable, totally disconnected sets E contain K-sets homeomorphic to themselves and that each such set E is almost a K-set in the sense that the identity map from E to E can be uniformly approximated by homeomorphisms of E onto K-sets in E. More precisely, the two theorems are as follows:

THEOREM A. *Let* G *be a nondiscrete, locally compact, abelian* T_0-*group, and let* E *be an independent, nonvoid, compact, metrizable, totally disconnected subset of* G. *Then there exist a metric space* $C_\lambda(E, E)$ *of continuous functions from* E *to* E, *complete in the uniform topology and containing the identity map from* E *to* E, *and a subset* H *of the first category in* $C_\lambda(E, E)$ *with the property that each* $f \in C_\lambda(E, E) \setminus H$ *maps* E *homeomorphically onto a* K-*set.*

THEOREM B. *Let* G *be a nondiscrete, locally compact, abelian* T_0-*group, and suppose that the torsion subgroup of* G *is at most countable. Let* E *be a subset of* G *homeomorphic to Cantor's ternary set. Then the set* C(E, E) *of continuous functions from* E *to* E *with the uniform topology contains a set* H *of the first category with the property that each* $f \in C(E, E) \setminus H$ *maps* E *homeomorphically onto a* K-*set.*

Definitions and Notation. In all that follows, G denotes a locally compact abelian T_0-group with character group X. We write C(E, T) for the set of continuous functions from E to the unit circle T in the complex plane.

A nonvoid compact subset E of G is called a K-*set* if $X \mid E$, the set of restrictions to E of continuous characters of G, is uniformly dense in C(E, T). We remind the reader that a K-set consists solely of independent elements of infinite order and that a nonvoid finite independent set is necessarily a K-set. (A finite subset $\{x_1, \cdots, x_k\}$ of G is called *independent* if the relation $x_1^{n_1} \cdots x_k^{n_k} = e$, where e is the identity of G and the exponents n_j are integers, implies that all the exponents n_j are zero. An infinite subset of G is called *independent* if every finite subset of it is independent. The void set is independent.)

Remarks. (a) We prove both theorems by using an argument whose original form is due to R. Kaufman [2]. A modification of Kaufman's argument given by Y. Katznelson [1, pp. 184-185] has been adapted for use here and in the related paper [3].

(b) Suppose that E consists of a convergent sequence together with its limit point x. Then each homeomorphism of E into E must map x to itself. Thus, the set of homeomorphisms of E into E is not dense in C(E, E). This example and a little further thought show that, in order to obtain a conclusion of the form "all

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