COMMON FIXED POINTS OF COMMUTING HOLOMORPHIC MAPS OF THE HYPERBALL

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1. INTRODUCTION

Let f and g be continuous functions on the closed unit disk of the complex plane, and suppose

(i) f and g are holomorphic in the open unit disk,

(ii) f and g map the closed disk into itself, and

(iii) f and g commute under the operation of function composition.

In [10], A. L. Shields showed that under these conditions, f and g have a common fixed point. D. J. Eustice [5] has extended this result to the polydisk in complex 2-space. W. M. Boyce [1] and J. P. Huneke [8] have independently given counterexamples to show that two continuous functions that commute and map the closed unit interval into itself need not have a common fixed point. In this paper, we extend the result of Shields to finite-dimensional inner-product spaces.

For a characterization of commuting polynomials, see [9] and [2]. For a more complete discussion of the history of problems concerning commuting maps, see [1].

2. HOLOMORPHIC IDEMPOTENTS ON THE UNIT BALL OF A HILBERT SPACE

We shall use the following notation.

(i) H is a Hilbert space (either finite-dimensional or infinite-dimensional) with inner product $\langle \cdot , \cdot \rangle$;

(ii) B is the unit ball of H, that is, $B = \{z \in H: \langle z, z \rangle < 1\}$, and $\overline{B}$ is the closure of B;

(iii) $\mathscr{H}(B)$ is the set of functions $f: B \to B$ that are holomorphic on B; and

(iv) if $f \in \mathscr{H}(B)$, then f^k is defined inductively for $k = 1, 2, \cdots$ by $f^1 = f$ and $f^{k+1} = f \circ f^k$.

The unit ball B of H is known to be a homogeneous domain (see [3] and [7]). That is, corresponding to each pair of points $u, v \in B$ there exists a function $L \in \mathscr{H}(B)$ such that $L(u) = v$, L is a one-to-one map of B onto B, and L^{-1} is holomorphic. Such a map is called a biholomorphic map of B onto B. For example, if we write u in the form $u = \alpha b$, where $\|b\| = 1$ and α is a complex number such that $|\alpha| < 1$, then the map L_u defined by

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