LIPSCHITZ FUNCTIONS AND BAD METRICS

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We explore conditions under which a metric space admits metrics that are in a certain sense smaller than the original metric. The question is related to the existence of certain Lipschitz functions, and the answers throw light upon some problems encountered by J. D. Stein [6], concerning the Lipschitz index of a metric space and the ideal structure of Lipschitz algebras.

1. BAD METRICS

Let (X, ρ) be a metric space. We say that a pseudometric σ on X (see [4]) is *smaller* than ρ on X if for each point x in X the quotient

$$\frac{\sigma(x, y)}{\rho(x, y)} \tag{1}$$

converges to zero as $\rho(x, y) \downarrow 0$, $y \in X$. We say that σ is *much smaller* than ρ on X if for each point x in X the quotient

$$\frac{\sigma(y, z)}{\rho(y, z)}$$

converges to zero as $\rho(x, y) \downarrow 0$ and $\rho(x, z) \downarrow 0$. A metric σ on X is *larger* than ρ if for each x in X the quotient (1) tends to infinity as $\rho(x, y) \downarrow 0$. Observe that the statements "σ is larger than ρ" and "ρ is smaller than σ" are not necessarily equivalent. They are equivalent if σ induces the same topology as ρ. Also, ρ is smaller than ρ if and only if (X, ρ) is a discrete topological space. If h(r) is a concave, increasing function, defined for $r \geq 0$, with $h(0) = 0$, then $h \circ \rho$ is a metric on X. The spaces (X, ρ) and $(X, h \circ \rho)$ are homeomorphic if the restriction of h to the image of ρ is continuous at 0. Thus there exist metrics on X that are larger than ρ and induce the same topology. On the other hand, it may happen that there are no metrics, or even nonzero pseudometrics, smaller than ρ. For instance, if σ is a pseudometric on Euclidean space $\mathbb{R}^n$ that is smaller than the Euclidean metric, then

$$\frac{|\sigma(x, z) - \sigma(y, z)|}{|x - y|} \leq \frac{\sigma(x, y)}{|x - y|},$$

hence the gradient $\nabla_x \sigma(x, z)$ vanishes identically as a function of x, for each fixed z, hence σ is identically zero.

We say that ρ is a *bad metric* on X if X admits a smaller metric that induces the same topology. We say that ρ is a *good* metric if X admits no smaller metric,

Received April 29, 1974.
This work was partially supported by NSF Grant No. GP33693X.

Michigan Math. J. 21 (1974).