A CHARACTERIZATION OF THE COMPLEX SPHERE

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1. INTRODUCTION

Let $P_{n+1}(C)$ denote the $(n+1)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1, and let $z_0, z_1, \cdots, z_{n+1}$ be a homogeneous coordinate system of $P_{n+1}(C)$. Let

$$Q_n = \left\{ (z_0, z_1, \cdots, z_{n+1}) \in P_{n+1}(C) \,\middle|\, \sum z_i^2 = 0 \right\}.$$

Then, with respect to the induced Kaehler structure, Q_n is an Einstein manifold with scalar curvature $n/2$, and it is complex analytically isometric to the Hermitian symmetric space $SO(n+2)/SO(2) \times SO(n)$. We call Q_n an n-dimensional *complex sphere*. In [3], the second author proved the following.

PROPOSITION. *Let* M *be an* n-*dimensional complete Kaehler submanifold immersed in* $P_m(C)$. *If the Ricci curvature of* M *is everywhere greater than* $n/2$, *then* M *is totally geodesic.*

The purpose of this paper is to prove the following theorem, which gives a local characterization of complex spheres.

THEOREM. *Let* M *be an* n-*dimensional Kaehler submanifold immersed in* $P_m(C)$. *If the Ricci curvature of* M *is everywhere equal to* $n/2$, *then* M *is locally* Q_n *in some* $P_{n+1}(C)$ *in* $P_m(C)$.

For notation and terminology, we follow [4], unless it is otherwise stated.

2. PRELIMINARIES

We prepare a brief summary of some basic facts. Details are found in [4].

Let M be an n-dimensional Kaehler submanifold immersed in $P_m(C)$. Let g, S, and ρ denote the Kaehler metric, the Ricci tensor, and the scalar curvature of M, respectively. If we denote by σ or A_α the second fundamental form of the immersion, then

$$S(X, Y) = \frac{n+1}{2} g(X, Y) - 2 \sum g(A_\alpha^2 X, Y), \tag{1}$$

$$\rho = n(n+1) - \|\sigma\|^2 . \tag{2}$$

Moreover, σ and A_α satisfy the differential equation

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