AN ELEMENTARY PROOF OF THE PICK-NEVANLINNA INTERPOLATION THEOREM

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1. INTRODUCTION

Let $\{z_1, \cdots, z_n\}$ be an n-tuple of distinct points in the open unit disk Δ. Let $\{w_1, \cdots, w_n\}$ be an n-tuple of complex numbers. The problem is to formulate a necessary and sufficient condition for the existence of a function f, analytic in Δ, bounded in modulus by 1, and such that $f(z_i) = w_i$ $(1 \leq i \leq n)$. Such an f is said to interpolate the sequences $\{z_i\}$ and $\{w_i\}$. The problem was originally solved by G. Pick [4] in 1916. His necessary and sufficient condition was that the n-by-n matrix

$$M = \left[\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j} \right] \quad (1 \leq i, j \leq n) \tag{1}$$

be positive semidefinite (nonnegative). R. Nevanlinna [2], [3] also solved the problem independently of Pick in 1919; however his conditions were rather implicit. He developed the following recursive relationship: If E is the set of analytic functions in Δ whose modulus is bounded by 1, then f is in E if and only if

$$f_1(z) = \frac{f(z) - f(z_1)}{1 - \overline{f(z_1)} f(z)} \Big/ \frac{z - z_1}{1 - \bar{z}_1 z}$$

is in E. In other words, f interpolates the n-tuples $\{z_i\}$ and $\{w_i\}$ $(1 \leq i \leq n)$ if and only if f_1 interpolates the (n - 1)-tuples

$$\{z_i\} \quad \text{and} \quad \left\{ \frac{w_i - w_1}{1 - \bar{w}_1 w_i} \Big/ \frac{z_i - z_1}{1 - \bar{z}_1 z_i} \right\} \quad (2 \leq i \leq n).$$

Repeating the process, we then obtain $f_2, f_3, \cdots, f_n$. Nevanlinna's theorem asserts that a necessary and sufficient condition for E to contain a function f with the property that $f(z_i) = w_i$ $(1 \leq i \leq n)$ is that the corresponding functions $f_1, f_2, \cdots, f_n$ belong to E. In 1956, B. Sz.-Nagy and A. Korányi [6] gave a proof of Pick's condition, using Hilbert-space techniques. In 1967, D. Sarason [5] gave a proof by means of operator theory. A host of others have considered similar problems for infinite sequences.

We shall give an elementary constructive proof that (1) is sufficient, and we shall show that the interpolating functions can be taken from a much smaller class than the analytic functions bounded by 1. Using these considerations, we also give a proof that (1) is necessary. Finally, in Section 3, we draw some consequences of these results.

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