FINITELY GENERATED FUCHSIAN GROUPS AND CHARACTER-AUTOMORPHIC NORMAL FUNCTIONS

E. Röding

Ch. Pommerenke [4] (Corollary 2) has shown that for every infinitely generated Fuchsian group there exists a character-automorphic function f(z) in $D = \{|z| < 1\}$ with

$$1 \leq \sup_{z \in D} (1 - |z|^2) f^\#(z) \leq K_0 < \infty,$$

where K_0 is an absolute constant. Here we use the notation

$$f^\#(z) = |f'(z)|/(1 + |f(z)|^2)$$

for the spherical derivative. We prove the following supplementary result.

THEOREM. *For every finitely generated Fuchsian group* Γ *there exists a nonconstant character-automorphic function* g(z) *with*

$$\sup_{z \in D} (1 - |z|^2) g^\#(z) \leq K_0 < \infty, \tag{1}$$

where K_0 *is an absolute constant.*

Proof. The case where Γ is finitely generated and of the second kind has been treated by Pommerenke in Section 3 of his paper [4]. Clearly, it suffices to consider the case where D/Γ is a compact Riemann surface. According to A. Marden [2], one can choose a conjugate group $\Gamma^* = \psi \circ \Gamma \circ \psi^{-1}$ such that there exists a fundamental region of Γ^* in D whose interior contains a circle K around 0 with radius $\rho > 0$ independent of Γ^*. There exists a single-valued potential function u on $R = D/\Gamma^*$ that has the singular behavior of $\log|z/(z - z_0)|$ near the points on R corresponding to 0 and some fixed point $z_0 \in K$ $(z_0 \neq 0)$, and is harmonic elsewhere. If $\tilde{u}$ denotes a conjugate harmonic of u, then the function $f = \exp(u + i\tilde{u})$ is a nonconstant character-automorphic function in D with respect to Γ^*.

Let ρ_1 denote a positive number such that $|z_0| < \rho_1 < \rho$, let

$$B = \{|z| < \rho\}, \quad \beta = \{|z| = \rho\}, \quad B_1 = \{|z| < \rho_1\}, \quad \alpha = \{|z| = \rho_1\},$$

let A denote the complement of B_1 on R, and let $u_0 = \log|z/(z - z_0)|$. The alternating method of Schwarz (see R. Nevanlinna [3, pp. 151-153]) requires the construction of functions u_n and v_n, harmonic in A and B, respectively, and with the boundary values

$$u_n = v_{n-1} + u_0 \quad \text{on } \alpha \ (v_0 \equiv 0), \qquad v_n = u_n - u_0 \quad \text{on } \beta. \tag{2}$$

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