AN APPLICATION OF UNIVERSAL ALGEBRA IN GROUP THEORY

Andreas Blass and Peter M. Neumann

Our purpose in this note is to give two proofs of the following result.

THEOREM. *Suppose that* G *is a group and* H *a subgroup which is the intersection of subgroups of finite index in* G. *Then* H *is not conjugate in* G *to a proper subgroup of itself.*

The first proof, the line of reasoning which first led to the discovery of the theorem, is a very natural application of standard facts of universal algebra. Since it has other applications, one of which we shall mention at the end of this note, we feel that the idea is worth recording even though the second proof is a direct group-theoretic argument which shows a little more.

First Proof. We begin by defining some concepts of universal algebra which are perhaps better known in the narrower context of group theory. A congruence ρ on an algebra A is said to have *finite index* if the quotient algebra A/ρ is finite; an algebra A is said to be *residually finite* if, for each pair of distinct elements $a, b \in A$, there is a congruence ρ of finite index such that $a \not\equiv b \pmod{\rho}$; and an algebra A is said to be *Hopfian* if every surjective endomorphism of A is an automorphism. Our proof is based on the following generalization of a group-theoretic theorem often attributed to A. I. Mal'cev (see [4, pages 116 and 415]).

LEMMA. *Finitely generated residually finite algebras are Hopfian.*

This lemma has been known for many years, but since proofs do not appear in the texts, we include a proof which is a little simpler than that of T. Evans [1].

Proof of the lemma. Suppose that A is a finitely generated, residually finite algebra, that ϕ is a surjective endomorphism of A, and that a and b are distinct elements of A; we must prove that $a\phi \neq b\phi$. By hypothesis, there is a congruence ρ of finite index on A such that $a \not\equiv b \pmod{\rho}$. Define another congruence by

$$\sigma := \{(x, y) \in A \times A \mid x\psi = y\psi \text{ for all homomorphisms } \psi\colon A \to A/\rho\}.$$

Since A/ρ is finite and A is finitely generated, there are only finitely many homomorphisms $\psi\colon A \to A/\rho$, and so σ, being the intersection of finitely many congruences of finite index, is itself a congruence of finite index on A. Furthermore, if ψ is such a homomorphism, then so is $\phi\psi$. Thus, if two elements of A are congruent modulo σ, then so are their ϕ-images. It follows that ϕ induces a homomorphism $\bar{\phi}\colon A/\sigma \to A/\sigma$ which is surjective since ϕ is. Because A/σ is finite, $\bar{\phi}$ must also be injective. This means that, if two elements of A are not congruent modulo σ, then neither are their ϕ-images and, *a fortiori*, these images are distinct. But a and b are clearly not congruent modulo σ, for they have different images under the canonical projection from A to A/ρ. Therefore, $a\phi \neq b\phi$. (This proof showed, in effect, that every congruence of finite index on a finitely generated algebra contains a fully invariant congruence of finite index.)

Received April 15, 1974.

Michigan Math. J. 21 (1974).