REMARKS ON THE INVARIANT-SUBSPACE PROBLEM

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1. The *invariant-subspace problem* for operators on Hilbert space is the question whether every bounded, linear operator on a separable, infinite-dimensional, complex Hilbert space $\mathscr{H}$ maps some (closed) subspace different from (0) and $\mathscr{H}$ into itself. In this note we prove three theorems, all of which are concerned with equivalent reformulations of this problem. The main tools employed are the Lomonosov technique and the theory of subdiagonalization of compact operators.

In what follows, $\mathscr{L}(\mathscr{H})$ will denote the algebra of all bounded, linear operators on $\mathscr{H}$. Moreover, all subalgebras of $\mathscr{L}(\mathscr{H})$ under consideration will be assumed to contain the identity operator $1_{\mathscr{H}}$. The lattice of invariant subspaces of a subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ will be denoted by $\operatorname{Lat}(\mathscr{A})$, and the algebra $\mathscr{A}$ will be called *transitive* if $\operatorname{Lat}(\mathscr{A}) = \{(0), \mathscr{H}\}$. The lattice of invariant subspaces of a single operator T will be denoted by $\operatorname{Lat}(T)$.

THEOREM 1.1. *Let $\mathscr{A}$ be a subalgebra of $\mathscr{L}(\mathscr{H})$. Then the following statements are equivalent:*

(1) *$\mathscr{A}$ is transitive.*

(2) *For every nonzero, quasinilpotent, compact operator K on $\mathscr{H}$, there exist an operator $A = A_K$ in $\mathscr{A}$ and a nonzero vector $x = x_K$ in $\mathscr{H}$ such that* $AKx = x$.

(3) *For every nonzero, nilpotent operator N on $\mathscr{H}$ of rank one, there exist an operator $A = A_N$ in $\mathscr{A}$ and a nonzero vector $x = x_N$ in $\mathscr{H}$ such that* $ANx = x$.

Proof. That (1) implies (2) follows from the basic Lomonosov theorem (see [1] and [2]): *if $\mathscr{A}$ is a transitive subalgebra of $\mathscr{L}(\mathscr{H})$ and K is a nonzero compact operator on $\mathscr{H}$, then there exist an operator $A = A_K$ in $\mathscr{A}$ and a nonzero vector $x = x_K$ such that* $AKx = x$. That (2) implies (3) is obvious; therefore we complete the proof by showing that (3) implies (1). Arguing contrapositively, we suppose that $\mathscr{M}$ is a subspace different from (0) and $\mathscr{H}$ that is invariant under $\mathscr{A}$. Let x_0 and y_0 be any two nonzero vectors in $\mathscr{H}$ such that y_0 belongs to $\mathscr{M}$ and x_0 is orthogonal to $\mathscr{M}$, and let N be the (unique) nilpotent operator of rank one that maps x_0 to y_0. Then the range of N is contained in $\mathscr{M}$, and it follows that for each A in $\mathscr{A}$ the range of AN is contained in $\mathscr{M}$. Thus $NAN = 0$ and $(AN)^2 = 0$ for every A in $\mathscr{A}$, from which it follows that there cannot exist an operator A_N in $\mathscr{A}$ and a nonzero x_N in $\mathscr{H}$ satisfying the equation $A_N N x_N = x_N$. This contradicts (3), and thus completes the proof.

We remark that one may argue independently of the Lomonosov theorem to show that conditions (1) and (3) above are equivalent. The reader may supply the details of this argument himself.

COROLLARY 1.2. *Let A be an operator in $\mathscr{L}(\mathscr{H})$. Then A has a nontrivial invariant subspace if and only if there exists a nonzero compact operator K such that $p(A)K$ is quasinilpotent for every polynomial p.*

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