PERIODIC HOMEOMORPHISMS OF THE 3-SPHERE AND RELATED SPACES

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1. INTRODUCTION

All objects in this paper are in the PL category. Let h be a periodic homeomorphism of a space M. The cyclic group generated by h shall be denoted by $\langle h \rangle$. Two actions of $\langle h \rangle$ and $\langle h' \rangle$ on M are said to be *conjugate* if there exists a homeomorphism t of M such that $\langle tht^{-1} \rangle = \langle h' \rangle$. In this case, h and h' are called *weakly equivalent*. If $tht^{-1} = h'$, then h and h' are said to be *equivalent*.

E. E. Moise [11] and F. Waldhausen [17] have shown that up to weak equivalences, the 3-sphere S^3 admits exactly one orientation-preserving homeomorphism of even period with nonempty fixed-point set (see P. A. Smith [15] and Kim [4] for alternative proofs). In the present paper, we show that up to weak equivalences S^3 admits exactly one orientation-reversing homeomorphism of period 4k. It follows that there are exactly four Z_4-actions on S^3, up to conjugation (see P. M. Rice [13] for free actions and Kim [4] for semi-free actions). Therefore, all Z_{2n}-actions $(n \leq 2)$ on S^3 are classified (for Z_2-actions, see [8], [9], and [17]). We show further that no lens space L(p, q) $(p > 2)$ admits an orientation-reversing homeomorphism of period n for all $n \neq 4$. We also discuss some free involutions on a lens space L(p, q).

Let h be a homeomorphism of period n on L = L(p, q). Then there exists a homeomorphism $\bar{h}$ of $L/\langle h^k \rangle$, uniquely determined by h, such that $\bar{h}g = gh$, where $g: L \to L/\langle h^k \rangle$ is the orbit map generated by $\langle h^k \rangle$. We call $\bar{h}$ the homeomorphism on $L/\langle h^k \rangle$ induced by h. We say that h is *sense-preserving* if $h_\#$ induces the identity on $H_1(L)$. We shall denote the fixed-point set of h by Fix(h). Note that if h is orientation-reversing, then n must be even, and $\mathrm{Fix}(h) \neq \emptyset$ by the Lefschetz fixed-point theorem.

2. ACTIONS ON S^3

Consider S^3 as a subset of C^2, defined by $\{(z_1, z_2) \in C^2 \mid z_1 \bar{z}_1 + z_2 \bar{z}_2 = 1\}$. Define an orientation-reversing homeomorphism T of S^3 by $T(z_1, z_2) = (\omega z_1, \bar{z}_2)$, where $\omega = e^{2\pi i/n}$ and n is even. We call T the standard homeomorphism (of period n). Remark 2.1 may be helpful in elucidating the meaning of Theorem 2.2.

Remark 2.1. Let ϕ be an orientation-preserving homeomorphism of period n on S^3 and with $\mathrm{Fix}(\phi) \neq \emptyset$. It is known [11] that $\mathrm{Fix}(\phi)$ is a simple closed curve. By Waldhausen [17], $\mathrm{Fix}(\phi)$ is unknotted for n = 2, and it is unknotted for n = 2k for all k. A well-known conjecture, due to P. A. Smith, asserts that $\mathrm{Fix}(\phi)$ is unknotted for all n (see S. Eilenberg [1]). It can be seen that the fixed-point set of each orientation-reversing periodic homeomorphism on S^3 consists of two points. In

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