BLASCHKE PRODUCTS WITH DERIVATIVE IN H^p AND B^p

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1. INTRODUCTION

Let U denote the open unit disc in the complex plane. The Hardy class H^p $(0 < p < \infty)$ consists of all functions analytic in U for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p \, dt$$

is finite. The class B^p $(0 < p < 1)$ consists of all functions analytic in U for which

$$\|f\|_{B^p} = \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f(re^{it})| \, (1 - r)^{1/p-2} \, dt \, dr$$

is finite. See [4] for a general discussion of H^p, and see [5] for basic properties of B^p. We note here the well-known result that $H^p \subset B^p$ for all p in the interval (0, 1).

A function analytic and bounded in U is said to be an inner function if its boundary values have modulus 1 almost everywhere. Every inner function ϕ has a factorization $\phi = mbs$, where m is a monomial, b is a Blaschke product with zeros $\{a_n\}$ $(a_n \neq 0)$, and s is a singular inner function; that is,

$$b(z) = \prod_n \frac{\bar{a}_n}{|a_n|} \frac{a_n - z}{1 - \bar{a}_n z},$$

where $0 < |a_n| < 1$ and $\sum (1 - |a_n|) < \infty$, and

$$s(z) = \exp\left[- \int_0^{2\pi} [(e^{it} + z)/(e^{it} - z)] \, d\mu(t) \right],$$

where μ is a finite positive singular measure.

In [2], J. G. Caughran and A. L. Shields raised the question whether there exists a singular inner function s whose derivative s' is in $H^{1/2}$. In [3], M. R. Cullen showed among other things that the derivative of every singular inner function lies in B^p for all p $(0 < p < 1/2)$, and he conjectured that the derivative of no singular inner function lies in $B^{1/2}$. H. A. Allen and C. L. Belna disproved this conjecture by giving examples in [1] of singular inner functions with derivatives in B^p $(0 < p < 2/3)$. In this paper, we shall investigate the corresponding properties for Blaschke products. Since the derivative of each finite Blaschke product is obviously in every H^p-space, we need only consider infinite Blaschke products.

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