ON THE HESSIAN OF A FUNCTION AND THE CURVATURES OF ITS GRAPH

Robert C. Reilly

INTRODUCTION

In a well-known paper [3], S. S. Chern constructs certain complicated integrands, denoted by B_{m-h} (see [3, p. 84]), on a nonparametric hypersurface $x_{m+1} = f(x_1, \cdots, x_m)$ in $\mathbb{R}^{m+1}$. The purpose of this note is to interpret these forms in two ways. First, we show that they are closely related to the elementary invariants of the Hessian matrix $\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$. This is proved in two cases by Chern in [3] and by H. Flanders in [4]. For our second interpretation, valid in the cases when h is even, we require a concept of a Ricci tensor of order q in the theory of q-sectional curvature of J. A. Thorpe [10].

Several other results are scattered through the note. For example, we give a formula for the kth mean curvature function of a nonparametric hypersurface; it generalizes the well-known formula

$$m\sigma_1 = \sum_{j=1}^{m} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_j}\Big/\left(1 + \sum_{k=1}^{m}\left(\frac{\partial f}{\partial x_k}\right)^2\right)^{1/2}\right).$$

1. THE INVARIANTS OF A SYMMETRIC TRANSFORMATION

In this section, we review some elementary facts. Let us consider a symmetric linear transformation A: $V \to V$, where V is an m-dimensional inner-product space. We denote the eigenvalues of A by $\lambda_1, \lambda_2, \cdots, \lambda_m$.

1.1 *Definition.* If $0 \leq q \leq m$, then *the* q-*th invariant* $S_q(A)$ *of* A is the qth elementary symmetric function of the numbers $\lambda_1, \cdots, \lambda_m$. That is,

$$S_q(A) = \sum_{1 \leq i_1 < \cdots < i_q \leq m} \lambda_{i_1} \cdots \lambda_{i_q}.$$

Furthermore, the q*th Newton transformation* $T_q(A)$ associated with A is

$$T_q(A) = S_q(A) I - S_{q-1}(A) \cdot A + \cdots + (-1)^q A^q.$$

For the sake of convenience, we gather the principal facts concerning $S_q(A)$ and $T_q(A)$ into a single proposition.

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