INTERPOLATION AND UNAVOIDABLE FAMILIES OF MEROMORPHIC FUNCTIONS

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The well-known Weierstrass factorization theorem says that given a sequence $\{a_n\}$ of complex numbers with no finite limit point, one can always construct an entire function f that vanishes precisely at the points a_n, where multiple occurrences of an a_n correspond to zeros of the corresponding multiplicity. It is natural to ask whether it is possible to construct an entire function whose zeros and one-points are prescribed. Now, by the well-known interpolation theorem for entire functions (see [6, p. 298]), given two sequences $\{a_n\}$ and $\{b_n\}$ that are disjoint and have no finite limit point, one can find an entire function f such that $f(a_n) = 0$ and $f(b_n) = 1$ for all n. But this does not answer the question, since it is conceivable that such an f must have other zeros or one-points. To see that this unpleasant possibility may actually arise, let $\{a_n\}$ be a finite nonempty set of cardinality A, and let $\{b_n\}$ be one of cardinality $B \neq A$. If a suitable f were to exist, it would omit 0, 1, and ∞ in a neighborhood of ∞, and would therefore have to be a polynomial, by Picard's Great Theorem; this is impossible, since $A \neq B$.

Our question is: For what pairs of disjoint sequences $\{a_n\}$ and $\{b_n\}$ without finite limit points can one construct an entire function f whose zero-sequence is exactly $\{a_n\}$ and whose one-sequence is exactly $\{b_n\}$? If this is possible, we call $(\{a_n\}, \{b_n\})$ the *zero-one set* of f. A more general form of this question was briefly studied by R. Nevanlinna in [5].

There are also *infinite* sequences $(\{a_n\}, \{b_n\})$ that are not zero-one sets. One way to see this was shown to us by J. Miles. By a result of A. Edrei [1, p. 277], an entire function with only real zeros and real ones has order at most 1. Since the exponent of convergence of the a-points of an entire function is no greater than the order of the function, we need only take $\{a_n\}$ and $\{b_n\}$ real and $\{a_n\}$, say, to have exponent of convergence greater than 1. By a slight variation of this argument, we can take the $\{a_n\}$ and $\{b_n\}$ arbitrarily sparse, so long as they lie on the real axis and each b_n is very close to some a_n; for this would force the derivative of an admissible entire function f to have order exceeding 1. This is impossible, since the order of f' equals the order of f. The same ideas show, for example, that there are three disjoint discrete sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ such that no pair of them forms a zero-one set.

In Theorem 1, we prove that to each sequence $\{a_n\}$ there corresponds a disjoint discrete sequence $\{b_n\}$ such that $(\{a_n\}, \{b_n\})$ is not the zero-one set of any entire function. We give two proofs of this result. The first proof is a bit complicated, but does not require deeper tools than Nevanlinna's First Fundamental Theorem. The second proof is due to J. Miles, whom we thank for allowing us to use it. It is simpler than the first proof, but does use Nevanlinna's Second Fundamental Theorem.

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