SUMMING SEQUENCES FOR AMENABLE SEMIGROUPS

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In a wide variety of settings, the special set $\sigma_n = \{0, 1, 2, \cdots, n-1\}$ enters into computations involving the additive semigroup Z^+ of nonnegative integers. In this paper, we identify the significant mathematical properties of the sequence $\{\sigma_n : n = 1, 2, \cdots\}$, and we show that if G is a countable, cancellative, amenable semigroup, then there exists a sequence $\{S_n\}$ of finite subsets of G possessing exactly these properties. In Section 3, we examine some examples and obtain miscellaneous properties.

1. PRELIMINARIES

Let G be a semigroup, and let m(G) denote the Banach space of bounded, real-valued functions on G endowed with the supremum norm

$$\|f\|_\infty = \sup\{|f(g)| : g \in G\}.$$

We shall also be interested in the subspace $\ell_1(G)$ consisting of the functions f in m(G) with finite ℓ_1-norm $\|f\|_1 = \sum \{|f(g)| : g \in G\} < \infty$. Endowed with the convolution

$$(f_1 * f_2)(g) = \sum \{f_1(h') f_2(h'') : h'h'' = g\},$$

$\ell_1(G)$ is a real Banach algebra.

A *weight* on G is a nonnegative function ϕ in $\ell_1(G)$ having finite support and such that $\|\phi\|_1 = 1$. A *simple weight* on G is a weight ϕ that is constant on its support; that is, ϕ is a simple weight provided $\phi = |A|^{-1}\chi_A$, where A, $|A|$, and χ_A denote the support of ϕ, the number of elements in the support, and its characteristic function. We denote the collection of all weights by Φ, the collection of all simple weights by Φ_s. For simplicity, given a g in G, we denote by g the simple weight with support $\{g\}$.

A *mean* on G is a real linear functional Λ on m(G) such that for each f in m(G),

$$\inf\{f(g) : g \in G\} \leq \Lambda(f) \leq \sup\{f(g) : g \in G\}.$$

Clearly, a mean Λ is a positive linear functional such that $\Lambda(1) = 1$, where 1 denotes the function $1(g) = 1$ for all g in G. If g is in G and f is in m(G), then ${}^g f$ and f^g are functions in m(G) defined by the equations

$${}^g f(h) = f(gh) \quad \text{and} \quad f^g(h) = f(hg),$$

respectively. A mean Λ on G is said to be *left* [*right*] *invariant* if $\Lambda({}^g f) = \Lambda(f)$ [if $\Lambda(f^g) = \Lambda(f)$] for all g in G and all f in m(G). Finally, G is said to be *amenable* if a left invariant mean and a right invariant mean exist on G.

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