SEMIGROUPS WITH IDENTITY ON E^3

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Let M be a semigroup with identity on E^3, and let G be the maximal connected subgroup containing 1. It is well known that G is a three-dimensional Lie group and an open subset of M. In this paper, we show that if G has a nontrivial compact subgroup, then the boundary of G contains an idempotent. This result is a partial answer to a question posed by P. Mostert and A. Shields [13].

Let L be the boundary of G, and let S be the closed subsemigroup $G \cup L$. Any action of a subgroup of G on M, S, or L (an ideal of S) will be the obvious one via the semigroup multiplication in M. We assume that G contains a nontrivial compact subgroup C. It follows that C is isomorphic to the multiplicative group of complex numbers of norm one [12]. Also, each of the sets

$$F_1 = \{x \in M \mid xC = \{x\}\} \quad \text{and} \quad F_2 = \{x \in M \mid Cx = \{x\}\}$$

is a closed subset of M that is homeomorphic to E^1 [10]. If x is a point of M not in F_1, then xC, the right C-orbit through x, is homeomorphic to C. A similar statement is true regarding F_2 and left C-orbits. Because the closure of each G-orbit in L is a one-sided ideal in S, we may assume that no G-orbit in L is compact.

The following lemma implies that for each x in L,

$$\dim xG = 1 \Rightarrow x \in F_1 \cap L \quad \text{and} \quad \dim Gx = 1 \Rightarrow x \in F_2 \cap L.$$

Thus $x \in L \setminus (F_1 \cup F_2) \Rightarrow \dim Gx = \dim xG = 2$.

LEMMA 1. *If* xG *is a one-dimensional* G-*orbit in* L *that contains a subset* K *that is homeomorphic to a circle, then* xG = K.

Proof. Let P be a one-parameter subgroup of G such that xP = xG, and let h: $P \to xP$ be the map h(p) = xp. If h is not one-to-one, then h(P) = K. Suppose that h is one-to-one, and that $xP \neq K$. We shall reach a contradiction. The inverse of K under h cannot be compact. There exists a sequence $\{p_i\}$ in P such that $\{p_i\}$ has no convergent subsequence, such that for each i, $h(p_i) \in K$, and such that $h(p_i) \to k$ in K. Let h(p) = k, and let I be any finite closed interval about p in P. Clearly, h(I) is an arc with k in its interior, and $h(I) \cap K$ contains no subarc with k in its interior.

The P-orbit xP is locally homeomorphic to $Z \times A$, where Z is a zero-dimensional subset of xP, and A is an arc [7]. Thus we may assume that $Z \times A$ is a neighborhood of k in xP that contains an arc A_1 in K about k and an arc A_2 in h(I) about k. For i = 1, 2, the projection of A_i onto Z is a connected subset of Z containing k; hence A_1 and A_2 are both contained in the same fibre $\{k\} \times A$. This implies that $A_1 \cap A_2$ is an arc about k, contrary to the results of the previous paragraph.

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