HYPERINVARIANT SUBSPACES VIA TOPOLOGICAL PROPERTIES OF LATTICES

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1. INTRODUCTION

We denote by $\mathscr{H}$ a fixed, separable, infinite-dimensional, complex Hilbert space, and by $\mathscr{L}(\mathscr{H})$ the algebra of all (bounded, linear) operators on $\mathscr{H}$. A (closed) subspace of $\mathscr{H}$ will be said to be *hyperinvariant* for an operator T on $\mathscr{H}$ if it is an invariant subspace for every operator that commutes with T. The question whether every operator on $\mathscr{H}$ has a nontrivial hyperinvariant subspace is one of the most difficult problems in the theory of invariant subspaces.

Our principal objective in the present paper is to derive the existence of nontrivial hyperinvariant subspaces for operators whose lattices of invariant subspaces have a certain topological property (Theorem 2.2). Subsequently, we apply this result to prove that an operator whose invariant-subspace lattice satisfies a certain purely lattice-theoretic condition has a nontrivial hyperinvariant subspace (Theorem 3.1).

For each T in $\mathscr{L}(\mathscr{H})$, we shall denote by Lat T the lattice of invariant subspaces of T. We assume that Lat T is equipped with the (relative) topology induced by the following metric on the collection of all subspaces of $\mathscr{H}$. If $\mathscr{M}$ and $\mathscr{N}$ are subspaces, and $P_{\mathscr{M}}$ and $P_{\mathscr{N}}$ denote the (orthogonal) projections onto $\mathscr{M}$ and $\mathscr{N}$, respectively, then the distance between the subspaces $\mathscr{M}$ and $\mathscr{N}$ is defined by $\theta(\mathscr{M}, \mathscr{N}) = \|P_{\mathscr{M}} - P_{\mathscr{N}}\|$. The study of the topological properties of Lat T considered as a metric space under the metric θ was initiated in [2]. There it was proved that if $\mathscr{M}$ is an inaccessible point of Lat T (that is, if the arcwise connected component of $\mathscr{M}$ in Lat T is the singleton $\{\mathscr{M}\}$), then $\mathscr{M}$ is a hyperinvariant subspace for the operator T. In particular, if $\mathscr{M}$ is an isolated point of Lat T, then $\mathscr{M}$ is a hyperinvariant subspace for T. An interesting consequence of this is the following result proved in [5] by different methods, and involving only a lattice-theoretic condition. If $\mathscr{M}$ is a pinch point of Lat T (that is, if $0 \neq \mathscr{M} \neq \mathscr{H}$ and $\mathscr{M}$ is comparable with every subspace in Lat T), then $\mathscr{M}$ is a nontrivial hyperinvariant subspace of T (proof: $\theta(\mathscr{M}, \mathscr{N}) = 1$ for every $\mathscr{N}$ in Lat T different from $\mathscr{M}$). A generalization of this result appeared in [3] and reads as follows. If Λ is a countable subset of Lat T such that every $\mathscr{M}$ in (Lat T) $\setminus \Lambda$ is comparable with each subspace in Λ, then every $\mathscr{M}$ in Λ is a hyperinvariant subspace for the operator T. These results provide a point of departure for the present note, and as will be seen later, all of them are easy corollaries of our main result (Theorem 2.2).

The motivating idea of this note is to find interesting and useful topological conditions on the invariant-subspace lattice of an operator T in order to guarantee that T has nontrivial hyperinvariant subspaces. Since there may exist operators T in $\mathscr{L}(\mathscr{H})$ such that Lat T = $\{(0), \mathscr{H}\}$, one might question the utility of such hyperinvariant-subspace theorems; but it should be remembered that there are several interesting classes of operators (for example, the compact operators) for which the

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