A GENERALIZATION OF A THEOREM OF KAPLANSKY AND RINGS WITH INVOLUTION

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I. Kaplansky has shown that if R is a semisimple ring each of whose elements is *power-central* (that is, if for each element x of R, there exists a positive integer n(x) such that $x^{n(x)}$ is in the center Z of R), then R is a commutative ring.

In this paper, we generalize Kaplansky's theorem to a ring with involution each of whose *symmetric* elements is power-central. We show that if the ring R has no nil right ideals, then all norms xx^* and all traces $x + x^*$ in the ring are central elements. If we weaken the assumption of no nil right ideals and assume merely that R has no nil ideals, the conclusion still holds, provided the least positive exponent n(s), for which $s^{n(s)} \in Z$, remains bounded as s ranges over the subset of symmetric elements in R (see Theorem 4).

Since semisimple rings R have no nil right ideal other than 0 (for brevity, we refer to them as rings *with no nil right ideal*), the first part of Theorem 4 generalizes Kaplansky's theorem.

I. N. Herstein has established the following extension of Kaplansky's theorem: A ring R with no nil ideal all of whose elements are power-central is a commutative ring [4]. The second part of Theorem 4 generalizes Herstein's theorem in the case of *bounded* exponents. The conclusion is the best one could expect; for if R consists of the 2-by-2 matrices over a field of characteristic 2, its symmetric elements under the involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ are the matrices $\begin{pmatrix} a & b \\ c & a \end{pmatrix}$, which are all square-central.

Whether the assumption "with no nil right ideal" in Theorem 4 can be replaced by its two-sided version "with no nil ideal" is an open question equivalent to a question of K. McCrimmon (see Section 4).

We break the proof of Theorem 4 into three steps. In Section 1 we prove the result for the case of division rings (see Theorems 1 and 2). In Section 2, we extend the results of Section 1 to certain *-prime rings (see Theorem 3), and in Section 3 we establish the general result by reduction to the preceding case. The author was inspired by Herstein's proof of [5, Theorem 3.2.2], and similar techniques are used. Finally, in Section 4, we give an example of a division ring all of whose symmetric elements are square-central, but not all of whose elements are central, and we conclude the paper with some open questions.

1. DIVISION RINGS

Some notation and conventions: Throughout this paper, R denotes a ring with involution in which

Received July 21, 1971, and November 10, 1972.

This research was supported by Grant A78676 of the NRC of Canada and by a summer fellowship of the Canadian Mathematical Congress (Branch at Université Laval). The author would like to thank the referee for many valuable suggestions.

Michigan Math. J. 20 (1973).