EXTREMAL PROBLEMS IN ARBITRARY DOMAINS

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1. INTRODUCTION

Let U be a domain in the extended complex plane $\mathbb{C}^*$, and let $H^\infty(U)$ be the uniform algebra of bounded analytic functions on U. Let L, $L_1, \cdots, L_m$ be linear functionals on $H^\infty(U)$, and let $b_1, \cdots, b_m$ be complex numbers. We are interested in the following "Pick-Nevanlinna" extremal problem:

(*) To maximize $\Re L(f)$, among all functions $f \in H^\infty(U)$ that satisfy the conditions $\|f\| \leq 1$ and $L_j(f) = b_j$ $(1 \leq j \leq m)$.

1.1 THEOREM. *Suppose* L, $L_1, \cdots, L_m$ *are continuous with respect to the norm* $\|\cdot\|_K$ *of uniform convergence on* K, *for some compact subset* K *of* U *that does not separate* ∂U. *Suppose also that* L *is not a linear combination of* $L_1, \cdots, L_m$, *and that there exists at least one competing function for* (*). *Then there exists a unique extremal function* G *for* (*). *The extremal function* G *has modulus* 1 *on the Shilov boundary of* $H^\infty(U)$, *and it can be extended analytically across each free analytic boundary arc of* U.

There is an extensive literature on extremal problems for analytic functions, in the case where U is bounded by analytic curves. For early references, see Z. Nehari's expository article [10]. The paper of A. J. Macintyre and W. W. Rogosinski [9] has a good introduction and bibliography, covering the case in which U is the unit disc, while the paper of H. L. Royden [11] deals with finite bordered Riemann surfaces. Arbitrary domains have been treated by S. Ya. Havinson [7] and S. D. Fisher [2], [3], and in spirit our work is based on that of Fisher.

The existence assertion of Theorem 1.1 follows immediately from the compactness of the family of competing functions. The uniqueness of the extremal function can be proved most easily by the technique of Fisher [2]. That the Ahlfors functions of arbitrary domains have the properties in Theorem 1.1 has already been established by Fisher [2], [3]. Our contribution is to extend these results to a more general class of extremal problems. The extension is not trivial, though, and the main point of the proof is the use of the separation theorem in Section 5 to reduce the problem (*) to a more tractable problem.

That the extremal function G has modulus 1 on the Shilov boundary can be converted into information concerning the cluster behavior of G. As a simple consequence of work in [5] and [6], we shall obtain the following corollary, which improves upon the corresponding results in [2] and [7].

1.2 COROLLARY. *If* w *is an essential boundary point of* U, *then the cluster set of the extremal function* G *at* w *either coincides with the closed unit disc, or else it lies on the unit circle.*

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