HOLOMORPHIC FUNCTIONS WITH LINEARLY ACCESSIBLE ASYMPTOTIC VALUES

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Let D and C denote the open unit disc and the unit circle. An arc $T \subset D$ *ends* at $z_0 \in C$ if $T \cup z_0$ is a Jordan arc. A holomorphic function f in D has *asymptotic value* w_0 at $z_0 \in C$ if there exists an arc $T \subset D$, ending at z_0, such that $f(z) \to w_0$ as $z \to z_0$ $(z \in T)$. The arc T is then an *asymptotic path* of f. If f maps T one-to-one onto a linear segment ending at w_0, then f has a *linearly accessible* asymptotic value at z_0. Let $A_L(f)$ denote the set of points at which f has linearly accessible values. G. R. MacLane [8, Theorems 3, 5, 7] has given several sufficient conditions for $A_L(f)$ to be dense on C. We shall give a necessary and sufficient condition for $A_L(f)$ to be dense on C.

Let S be a nonempty subset of D. For each r $(0 < r < 1)$, let the components of $S \cap \{z: r < |z| < 1\}$ be $S_\beta(r)$ $(\beta \in B)$. Let $d_\beta(r)$ be the diameter of $S_\beta(r)$, and let $d(r) = \sup_{\beta \in B} d_\beta(r)$. Clearly, d is a nonincreasing function of r. The set S *ends at points* of C if $d(r) \downarrow 0$ as $r \uparrow 1$.

If $w = f(z)$ is a nonconstant, holomorphic function in D, we denote by F the Riemann surface of f^{-1} (as a covering surface over the w-plane). Let p denote the projection from F onto the w-plane, and let $\tilde{f}$ be the one-to-one conformal map of D onto F, so that $f = p \circ \tilde{f}$. Corresponding to each set S in the w-plane, we denote by F_S the set of points of F lying over S.

MacLane's class $\mathscr{A}$ is the class of nonconstant holomorphic functions in D *that have asymptotic values at a dense subset of* C. *A function* f *belongs to class* $\mathscr{L}$ *if it is nonconstant and holomorphic in* D *and if for each* $r \geq 0$ *the level set* $\{z: |f(z)| = r\}$ *ends at points of* C. MacLane [7, Theorem 1] proved that $\mathscr{A} = \mathscr{L}$. We now state our main result.

THEOREM 1. *Let* f *be a nonconstant, holomorphic function in* D. *A necessary and sufficient condition for* $A_L(f)$ *to be dense on* C *is that there exists a line* K *in the* w*-plane such that the set* $\tilde{f}^{-1}(F_K)$ *ends at points of* C.

REMARKS. 1. In the notation of this paper, we can restate the assertion $\mathscr{A} = \mathscr{L}$ as follows. A necessary and sufficient condition for a nonconstant holomorphic function f to belong to class $\mathscr{A}$ is that the set $\tilde{f}^{-1}(F_{|w|=r})$ ends at points of C, for each $r \geq 0$. From this restatement it is clear that the condition of Theorem 1 for lifting lines is analogous to MacLane's condition expressed in $\mathscr{A} = \mathscr{L}$ for lifting circles.

2. In proving Theorem 1, we shall prove that a necessary condition for $A_L(f)$ to be dense on C is that the set $\tilde{f}^{-1}(F_K)$ ends at points of C *for every line* K *in the* w*-plane. Hence, if the set* $\tilde{f}^{-1}(F_K)$ *ends at points of* C *for one line* K *in the* w*-plane, then* $A_L(f)$ *is dense on* C *and hence the set* $\tilde{f}^{-1}(F_K)$ *ends at points for every line* K *in the* w*-plane.*

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