VECTOR-VALUED ANALYTIC FUNCTIONS

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In this note we show that a function f defined on a real domain and taking values in the dual of a Fréchet space is analytic if it is weakly analytic. A counterexample shows that this is not generally true for functions taking values in a Fréchet space.

We apply this result to a consideration of partially analytic distributions, and to a theorem of T. Kotake and M. S. Narasimhan on the behavior of elliptic partial differential operators with analytic coefficients.

First, let us fix some notational conventions. We shall denote by $\mathcal{O}$ the space of germs of functions holomorphic in a neighborhood of the point 0 in the complex plane $\mathbf{C}$, and by $\mathcal{P}$ the space of formal power series in z. By $\mathcal{H}(B_r)$ we shall mean the space of holomorphic functions defined on the disc $B_r = \{z \in \mathbf{C}: |z| < r\}$. Then $\mathcal{O}$ is a (generalized) LF-space; $\mathcal{O} = \bigcup_{r>0} \mathcal{H}(B_r)$.

Our first theorem could be proved directly by means of the Baire Category Theorem. However, we prefer to use the theorem of Grothendieck that if $T: E \to F$ is a closed linear map, E is a Fréchet space, and $F = \bigcup F_k$ is an inductive limit of a sequence of Fréchet spaces, then there exists an N such that T maps E into F_N continuously. (See Theorem 3 and the remark at the end of this note.)

THEOREM 1. *Let* E *be a Fréchet space and* E' *its strong dual. If* $\{f_j\}$ *is a sequence of elements of* E' *such that, for each* $u \in E$, *the sequence* $\{|\langle f_j, u\rangle|^{1/j}\}$ *is bounded, then there exists an* $\varepsilon > 0$ *such that* $\{\varepsilon^j f_j\}$ *is strongly bounded.*

Proof. For each $u \in E$, the element $Tu = \sum_{j=0}^{\infty} \langle f_j, u\rangle z^j$ belongs to $\mathcal{O}$. Since the composite map $E \xrightarrow{T} \mathcal{O} \to \mathcal{P}$ is clearly continuous, T is closed. Grothendieck's theorem implies that T maps E continuously into $\mathcal{H}(B_r)$ for some $r > 0$, which implies that $|\langle f_j, u\rangle| \leq C(2/r)^j$ for each $u \in E$, with r independent of u. Our theorem follows, with $\varepsilon = r/2$, by the uniform-boundedness theorem.

THEOREM 2. *Let* Ω *be a region in* $\mathbb{R}^n$, *and suppose* $f: \Omega \to E'$ *is a function such that, for each* u *in the Fréchet space* E, *the map* $x \to \langle f(x), u\rangle$ *is analytic. Then* f *is strongly analytic.*

Proof. If $x_0 \in \Omega$, then for each $u \in E$, $\frac{1}{h}\langle f(x_0 + h\varepsilon_j) - f(x_0), u\rangle$ converges to $(\partial/\partial x_j)\langle f(x), u\rangle|_{x_0}$ as $h \to 0$, where $\{\varepsilon_1, \cdots, \varepsilon_n\}$ is the standard orthonormal basis of $\mathbb{R}^n$. The uniform boundedness theorem implies that there is an $f_j \in E'$ such that

$$\langle f_j, u\rangle = \frac{\partial}{\partial x_j}\langle f(x), u\rangle|_{x_0} \qquad \text{for all } u \in E.$$

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