COMPACTNESS PROPERTIES OF TOPOLOGICAL GROUPS

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In a topological group, an element is called *bounded* if its conjugate class is relatively compact. The concept of bounded elements is useful in the study of the structure of locally compact groups. Many results concerning bounded elements have been established by V. I. Ušakov (see [6], [7], [8], [9]) and J. Tits [5]; we mention two of them for later use in this paper.

THEOREM A (V. I. Ušakov). *Let* G *be a totally disconnected, locally compact group. If a relatively compact subset* A *of* G *is invariant under all inner automorphisms of* G, *and if each of its elements belongs to a compact subgroup of* G, *then the closed subgroup generated by* A *is a compact, normal subgroup of* G.

THEOREM B (J. Tits). (a) *If* G *is a projective limit of Lie groups, then the set* B(G) *of all bounded elements of* G *is closed in* G.

(b) *Let* G *be an analytic group without compact, normal subgroups except the identity subgroup. Then*

(i) *the identity component* $B_0(G)$ *of* B(G) *is a vector group,*

(ii) *if* Z(G) *denotes the center of* G, *then* $B(G) = B_0(G)\,Z(G)$, *and*

(iii) *if* α *is a bounded automorphism on* G (*that is, if the set* $\{\alpha(g)\,g^{-1} \mid g \in G\}$ *is relatively compact*), *then there exists an element* g *in* B(G) *such that the inner automorphism* I_g *on* G *induced by* g *is equal to* α.

The set B(G) of bounded elements of a topological group G actually forms a characteristic subgroup of G; we shall call it the *bounded part* of G. A group is called an $\overline{FC}$-*group* if all its elements are bounded. In this paper, we relax the condition of boundedness in three directions: (I) B(G) is open, (II) $\overline{B(G)} = G$, and (III) $G/\overline{B(G)}$ is compact.

Locally compact groups with open bounded parts will be discussed in Section 2, where we shall prove the following two results:

(a) *In a locally compact group, the bounded part is open if and only if there exists a compact invariant neighborhood of the identity.*

(b) *In a* σ*-compact, locally compact group, the bounded part is open if and only if it is of second category.*

Sections 3 and 4 are devoted to the study of locally compact groups with dense bounded parts. There we shall generalize some results on $\overline{FC}$-groups and suggest a structure theorem. The main results are as follows.

(c) *In a locally compact group with dense bounded part, the periodic part* (*the set of elements that are contained in compact subgroups*) *forms a closed characteristic subgroup whose factor group is a direct product of a vector group and a discrete, torsion-free, abelian group.*

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