THE MARX CONJECTURE FOR STARLIKE FUNCTIONS

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1. INTRODUCTION

In 1932, A. Marx [4] conjectured that for each fixed z_0 in the unit disc, the set of possible values of $f'(z_0)$ for all f in the class of starlike functions is contained in the set of all values of $k'(z)$ for $|z| \leq |z_0|$, where $k(z) = z/(1 - z)^2$ is the Koebe function. Marx showed that this must be true for $|z_0| \leq \sin \pi/8 = 0.382 \cdots$. R. M. Robinson [5], [6] improved the results of Marx, and more recently, P. L. Duren [1] proved that the conjecture holds for $|z_0| \leq 0.736 \cdots$.

In this paper we present a counterexample, which shows that the conjecture is in fact false. We also prove that every point on the boundary of the set of possible values is given by a function having at most two slits; this simplifies the problem considerably.

First, we must state precisely the form of the problem as we shall investigate it. Following Robinson [5], [6], we replace Marx's original formulation of the problem by the investigation of the domain of variability of $\log f'(z_0)$. This has the advantage of making the mappings involved univalent, at the expense of requiring care in keeping track of the proper branch of the logarithm.

2. THE MARX REGION

Let U denote the unit disc $\{z: |z| < 1\}$. Let $\mathscr{S}^*$ denote the class of *starlike functions*, that is, the class of all functions that are regular and univalent in U with $f(0) = 0$, $f'(0) = 1$, and that map U onto a domain starshaped with respect to the origin. For $z_0 \in U$, we define the *Marx region* for z_0 as

$$M(z_0) = \{w: w = \log f'(z_0),\ f \in \mathscr{S}^*\}. \tag{2.1}$$

The determination of the branch of the logarithm is fixed by the specification that $0 \in M(z_0)$ and that $f_t(z) = \frac{1}{t} f(tz)$ $(0 < t \leq 1)$. Then $f_1(z) = f(z)$, and since $f_t(z) = z + a_2 tz^2 + \cdots$, we may let $f_0(z) = z$. Each f_t is in $\mathscr{S}^*$; hence $\log f_t'(z_0)$ $(0 \leq t \leq 1)$ defines a path in $M(z_0)$ joining 0 to $\log f'(z_0)$.

The set $M(z_0)$ is bounded, because of the well-known bounds on $f'(z)$ for $f \in \mathscr{S}^*$. Since $\mathscr{S}^*$ is a normal family, $M(z_0)$ is closed. It is symmetric about the real axis; for if $f(z) = \sum a_n z^n$ is in $\mathscr{S}^*$, then $\sum \bar{a}_n z^n$ is in $\mathscr{S}^*$.

The set $M(z_0)$ depends only on $r = |z_0|$. To see this, note that $f \in \mathscr{S}^*$ implies $e^{-i\alpha} f(e^{i\alpha} z) \in \mathscr{S}^*$ for every real α. From this one easily shows $M(z_0) = M(|z_0|)$.

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