PRODUCT MANIFOLDS THAT ARE NOT NEGATIVE SPACE FORMS

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INTRODUCTION

In [2], it was shown that for every integer $k \geq 1$ there exist compact Riemannian manifolds M of dimension $4k$ and sectional curvature K $(-4 \leq K \leq -1)$ that do not admit a Riemannian metric with $K \equiv -1$. In this paper, we show by a different method that there exist many (noncompact) product manifolds M that admit a complete metric with sectional curvature $K \leq -1$, but admit no complete metric with $K \equiv -1$. For example, if B is a compact Riemannian manifold of dimension $n \geq 2$ and sectional curvature $K < 0$, then $M = R \times B \times S^1$, where S^1 and R denote the unit circle and real line, is such a manifold.

If M is a complete Riemannian manifold with sectional curvature $K \leq 0$, then M is a quotient H/D, where H is a complete, simply connected manifold with $K \leq 0$, and D is a properly discontinuous group of isometries of H. As in [3], one may construct the set $H(\infty)$ of points at infinity for H and define a limit set $L(D) \subset H(\infty)$ for the group D. If $M = H/D$ has curvature $K \leq -1$, then the limit set must be one of three types described in Theorem 1. If $\pi_1(M)$ is a nontrivial direct product, then the discussion of monic groups shows that $L(D)$ must be of the first type. If M admits a complete metric with $K \equiv -1$, and if $L(D)$ is of the first type, then by Proposition 4, $\pi_1(M)$ is isomorphic to $\pi_1(M')$, where M' is a complete flat manifold. Using warped products, we construct manifolds M with sectional curvature $K \leq -1$ such that $\pi_1(M)$ is a nontrivial direct product not isomorphic to $\pi_1(M')$ for any flat manifold M'. Our result has the defect that the curvature of M is not bounded below; but this defect is inherent in our method.

The necessary background results are developed in [1] and [3], and more detailed discussions of points at infinity, limit sets, monic groups, and warped products may be found there with proofs of our unproved assertions.

MANIFOLDS WITH SECTIONAL CURVATURE $K \leq 0$

Let H denote a complete, simply connected Riemannian manifold of dimension $n \geq 2$ and sectional curvature $K \leq 0$. Unit-speed geodesics γ and σ in H are *asymptotic* if there exists a number $c > 0$ such that $d(\gamma t, \sigma t) \leq c$ for all $t \geq 0$, where d denotes the Riemannian metric of H. The relation of being asymptotic is an equivalence relation on the geodesics of H (which shall always be assumed to have unit speed), and the equivalence classes are *points at infinity* for H. If $H(\infty)$ denotes the set of points at infinity, then the space $\overline{H} = H \cup H(\infty)$ with a natural topology is homeomorphic to the closed unit ball in R^n, and $H(\infty)$ is homeomorphic to the bounding sphere S^{n-1}.

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